Solving Linear Difference Equations with Constant Coefficients and Initial Conditions/Boundary Conditions

1. Consider again the linear operator

$$L(y(k)) = y(k+2) - 5y(k+1) + 6y(k).$$

Here the y sequence has been written using function notation. Alternatively, it could be written using sequence notation:

$$L(y_k) = y_{k+2} - 5y_{k+1} + 6y_k.$$

The equation $L(y_k) = 0$ is equivalent to the LDEwcc

$$y_{k+2} - 5y_{k+1} + 6y_k = 0. \tag{1}$$

(a) Write down the characteristic equation (auxiliary equation) for equation (1), and solve it.

(b) Write the formula for the general solution of (1) with two arbitrary constants.

(c) Determine the values of the constants that make the general solution satisfy the two initial conditions $y_0 = 0$ and $y_1 = 1$, and write the corresponding functional formula for y_k.

(d) Determine the values of the constants that make the general solution satisfy the two initial conditions $y_0 = 4$ and $y_1 = 5$, and write the corresponding functional formula for y_k.

2. **The Gambler's Ruin: The Case of a Fair Game**
A gambler repeatedly wagers \$1, winning \$1 with probability 1/2 and losing \$1 with probability 1/2. The gambler keeps betting until s/he goes broke ("ruin") or attains an amount in-hand of N dollars. Let R_k denote the probability that the gambler will ultimately be ruined if s/he starts with \$$k$. Then

$$R_k = \frac{1}{2}R_{k-1} + \frac{1}{2}R_{k+1} \tag{2}$$

for $0 < k < N$. The *boundary conditions* are: $R_0 = 1$ and $R_N = 0$.

(a) Rewrite equation (2) in the form

$$aR_{k+2} + bR_{k+1} + cR_k = 0.$$

(b) Write down the characteristic equation, and solve it.

(c) Write an equation for the general solution of equation (2) with two arbitrary constants.

(d) Use the boundary conditions to determine the values of the two constants, and write the functional formula for R_k. Is this result intuitively reasonable?

(e) Yet to be explored: the more interesting case where the gambler wins with probability p and loses with probability $q = 1 - p$:

$$R_k = pR_{k+1} + qR_{k-1}.$$