Solving Homogeneous Second-Order Linear Difference Equations with Constant Coefficients

Let's find functional formulas for the solutions of any LDEwcc of the form

$$a\, y_{k+2} + b\, y_{k+1} + c\, y_k = 0.$$

Here a, b, and c are constants, and $a \neq 0$ and $c \neq 0$ so that this LDE is of order 2.
We seek geometric sequences that are solutions. Let

$$y_k = r^k.$$

1. Show that if $r \neq 0$, then

$$a\, r^2 + b\, r + c = 0.$$

(Later we'll recover the case that arises when $r = 0$.) Goldberg calls this the "auxiliary equation" associated with the given LDEwcc. Nowadays it is more commonly called the associated "characteristic equation."

2. The equation for r is a quadratic equation. Write the formula for the solutions of this quadratic equation.

3. There are two distinct roots if the discriminant $b^2 - 4ac$ is not zero. Call the two roots r_1 and r_2. Then (see Goldberg's equation (3.39)), the general solution of the given LDEwcc is given by

$$y_k = A\, r_1^k + B\, r_2^k$$

where A and B are arbitrary constants. In other words, every solution of the given LDEwcc is given by this formula for some choice of the values of A and B. The case where $A = B = 0$ yields the solution in the case in paragraph 1 when $r = 0$.

a. If the discriminant $b^2 - 4ac > 0$, then the two roots are both real numbers—if *a, b*, and *c* are real. We are used to dealing with such numbers.

b. If the discriminant $b^2 - 4ac < 0$ and if *a, b,* and *c* are real numbers, then the two roots are of the form $x \pm yi$ where $i = \sqrt{-1}$ and $x = -\frac{b}{2a}$ and $y = \frac{\sqrt{4ac-b^2}}{2a}$. The solution formula still works in this case, but the numbers involved are complex. It is possible to rewrite the solution in a real-number form. Goldberg pp. 138-142 shows how to rewrite the solution in the form $y_k = A\,R^k \cos(k\theta + B)$ where *A*, *B*, and θ are real constants. I'll leave it for the serious student to learn how to do this.

4. If the discriminant $b^2 - 4ac = 0$, then the quadratic equation has only one root—called a double root. Let $r = -\frac{b}{2a}$, the solution in this case. Then (see Goldberg equation (3.43)), the general solution of the given LDEwcc is given by

$$y_k = (A + Bk)r^k$$

where *A* and *B* are arbitrary constants.

5. Goldberg Problems 3.3 #1: "Find the general solution of each of the following homogeneous difference equations:

(a) $y_{k+2} - y_k = 0$

(b) $2y_{k+2} - 5y_{k+1} + y_k = 0$

(c) $y_{K+2} + 2y_{k+1} + y_k = 0$

(d) $9y_{k+2} - 6y_{k+1} + y_k = 0$

(e) $3y_{k+2} - 6y_{k+1} + 4y_k = 0$

(f) $y_{k+2} + 6y_{k+1} + 25y_k = 0$"

6. Goldberg Problems 3.3 #2(b): "[F]ind a particular solution satisfying the initial conditions $y_0 = 0$ and $y_1 = 1$." Use the style of "Writing Mathematics" in writing your solution of this problem.