## Linear Difference Equations with Constant Coefficients

**3.1 Some Basic Theorems.**
Here are the main difference equations we have studied so far:

$$\begin{aligned} y_{n+1} &= y_n + d && \text{for } n = 0, 1, 2, \ldots \\ y_{n+1} &= r y_n && \text{for } n = 0, 1, 2, \ldots \\ y_{n+1} &= r y_n + d && \text{for } n = 0, 1, 2, \ldots \end{aligned}$$

In Goldberg's *Introduction to Difference Equations*, which favors the use of index $k$, these would be written

$$\begin{aligned} y_{k+1} - y_k &= d && \text{for } k = 0, 1, 2, \ldots \\ y_{k+1} - r y_k &= 0 && \text{for } k = 0, 1, 2, \ldots \\ y_{k+1} - r y_k &= d && \text{for } k = 0, 1, 2, \ldots \end{aligned}$$

We should be able to cope with these index changes.

The general form of a *linear* difference equation is [1, p. 121] (**write your answer here**)

A *linear difference equation with constant coefficients* ("LDEwcc") is an equation for a sequence $\langle y_k \rangle$ that may be written in the form

$$y_{k+d} + a_1 y_{k+d-1} + \cdots + a_{d-1} y_{k+1} + a_d y_k = g_k$$

typically for $k = 0, 1, 2, \ldots$. Here $a_d \neq 0$, and then this is said to be a LDEwcc of *order* $d$. (Some people say it is of "degree" $d$.) (Goldberg uses $n$ for the order, but we like to use $n$ sometimes for our time index, and besides, $d$ is constant in any particular application, so "$d$" is a better choice [but don't confuse it with the "$d$" in the equations at the top of the page].) The LDEwcc is *homogeneous* if the sequence $\langle g_k \rangle$ is identically zero; otherwise it is *nonhomogeneous*.

**Exercise:** Classify the three main equations we have studied as: (a) linear, or nonlinear; (b) homogeneous, or nonhomogeneous; (c) of order ________.

| | | | | |
|---|---|---|---|---|
| $y_{k+1} - y_k = d$ | for $k = 0, 1, 2, \ldots$ | $(a)$ | $(b)$ | $(c)$ |
| $y_{k+1} - r y_k = 0$ | for $k = 0, 1, 2, \ldots$ | $(a)$ | $(b)$ | $(c)$ |
| $y_{k+1} - r y_k = d$ | for $k = 0, 1, 2, \ldots$ | $(a)$ | $(b)$ | $(c)$ |

**Exercise:** Rewrite the following equations in standard LDE form and classify them as: (a) linear, or nonlinear; (b) homogeneous, or nonhomogeneous; (c) with constant coefficients, or with variable coefficient(s); (d) of order ________.

(1) $\Delta y_k = k + 1$

(2) $\Delta^2 y_k = 0$

(3) $(1/2)y_{n+1} + (1/2)y_{n-1} = y_n$

(4) $y_k = 3y_{k-1} + ky_{k-2}$

**Theorem 3.1.** If $y^{(1)}$ and $y^{(2)}$ are any two solutions of a LDEwcc, then so is $C_1y^{(1)} + C_2y^{(2)}$ for arbitrary constants $C_1$ and $C_2$.

**Theorem 3.2.** If $Y$ is a solution of a homogeneous LDEwcc and $y^*$ is a solution of the "complete equation," i.e., the corresponding equation with a nonzero right-hand side $g_k$, then their sum $Y + y^*$ is a solution of the complete equation.

**Theorem 3.3.** The general solution of the homogeneous first-order LDEwcc $y_{k+1} + a_1y_k = 0$ is given by $y = Y$ where $Y_k = C(-a_1)^k$ where $C$ is an arbitrary constant.
The general solution of the nonhomogeneous equation $y_{k+1} + a_1y_k = r_k$ is given by $Y_k = C(-a_1)^k + y_k^*$ where $y^*$ is any particular solution of the same equation.

**PROBLEMS 3.1.** Do problems 1af, 3af, and 4a on p. 127.

**1.** Identify $a_1, a_2, ...g_k$ if

(a) $y_{k+1} + 3y_k = 8$

(f) $2y_{k+3} - 5y_{k+2} + y_k = 2^k$

**3.** Show that each given $y^*$ is a particular solution of the corresponding difference equation in Problem 1.

(a) $y_k^* = 2$

(f) $y_k^* = -\frac{1}{3}2^k$

**4.** Find the general solution of Problem 1(a) by Theorem 3.3

## References


[1] Goldberg, Samuel, *Introduction to Difference Equations: with Illustrative Examples from Economics, Psychology, and Sociology*, Dover Publications, New York, 1958, 1986.