

Summation Notation

Sigma or summation is sometimes used to represent sums.

$$\sum_{j=1}^n t_j = t_1 + t_2 + t_3 + \cdots + t_n.$$

The “index” j here could be replaced by any unused letter.

Examples:

Sums and Recurrences

Let $s_n = \sum_{j=1}^n t_j = t_1 + t_2 + t_3 + \cdots + t_n$. Then $\langle s_n \rangle$ satisfies the recurrence

$$s_n = s_{n-1} + t_n \tag{1}$$

for $n = 1, 2, 3, \dots$ or, alternatively,

$$s_{n+1} = s_n + t_{n+1} \tag{2}$$

for $n = 0, 1, 2, \dots$ with initial condition $s_0 = 0$ in both cases.

Conversely, if we assume that

$$s_{n+1} = s_n + t_{n+1} \tag{3}$$

for $n = 0, 1, 2, \dots$ (or its alternative), then the solution (functional formula) is

$$s_n = s_0 + \sum_{j=1}^n t_j = s_0 + t_1 + t_2 + t_3 + \cdots + t_n. \tag{4}$$

Application 1: Arithmetic Sequences

Assume that the terms t_j are all the same: $t_j = d$, a constant, for every value of j . To match up the notation, also let $s_n = a_n$ for every n . Then equation (2) becomes

$$a_{n+1} = a_n + d$$

for $n = 0, 1, 2, \dots$, and its solution (functional equation) is given by

$$a_n = a_0 + dn$$

for $n = 0, 1, 2, \dots$. So, if $t_j = d$ for every j , $\sum_{j=1}^n t_j = \sum_{j=1}^n d = \underline{\hspace{2cm}}$.

Application 2: Sums of Arithmetic Sequences

Schoolboy Gauss's sum: $1 + 2 + 3 + \dots + 98 + 99 + 100$

Answer by Gauss's trick:

Let S_n be the sum of the first n terms of a general arithmetic sequence with common difference d :

$$S_n = c + [c + d] + [c + 2d] + \dots + [c + (n - 2)d] + [c + (n - 1)d].$$

$$S_n =$$

$$2S_n =$$

$$S_n =$$

(Notice that the n^{th} term is $c + (n - 1)d$, because the "first" term is $c + 0 \cdot d$.) You should have found above that

$$\sum_{k=0}^{n-1} [c + kd] = \sum_{j=1}^n [c + (j - 1)d] = cn + d(n - 1)n/2 = (c - d/2)n + (d/2)n^2, \quad (5)$$

a quadratic function of n .

Application 3: Quadratic Growth

Suppose that the differences between successive terms of a sequence $\langle q_n \rangle$ themselves form an arithmetic sequence $\langle d + en \rangle$:

$$q_{n+1} = q_n + d + en$$

for $n = 0, 1, 2, \dots$, where d and e are constants. (Alternatively, $q_n = q_{n-1} + d + e(n - 1)$.)

This pattern matches equation (3) with $s_n = q_n$ and $t_{n+1} = d + en$ (or, $t_n = d + e(n - 1)$). Thus, by equations (4) and (5),

$$q_n = q_0 + dn + e(n - 1)n/2.$$

This is a quadratic function of n .