MCS 121–Fall 2004: Chapter 4 exam answers

(1) $\dfrac{dv}{dL} = \dfrac{1}{2}\left(\dfrac{L}{2}+\dfrac{2}{L}\right)^{-1/2}\left(\dfrac{1}{2}-\dfrac{2}{L^2}\right)$, so $\dfrac{dv}{dL}=0 \Leftrightarrow \dfrac{1}{2}=\dfrac{2}{L^2} \Leftrightarrow L^2=4$. Since $L>0$, $L=2$. This gives a minimum by the first derivative sign-change test: $0<L<2 \Rightarrow \dfrac{1}{2}-\dfrac{2}{L^2} < \dfrac{1}{2}-\dfrac{2}{2^2}=0 \Rightarrow \dfrac{dv}{dL}<0$ and $L>2 \Rightarrow \dfrac{1}{2}-\dfrac{2}{L^2} > \dfrac{1}{2}-\dfrac{2}{2^2}=0 \Rightarrow \dfrac{dv}{dL}>0$.

(*Alternative tests*: (1) $v\to\infty$ as $L\to 0^+$ and as $L\to\infty$;

(2) $\dfrac{d^2v}{dL^2} = \dfrac{2\sqrt{\frac{L}{2}+\frac{2}{L}}(\frac{4}{L^3}) - (\frac{1}{2}-\frac{2}{L^2})(\frac{L}{2}+\frac{2}{L})^{-1/2}(\frac{1}{2}-\frac{2}{L^2})}{4(\frac{L}{2}+\frac{2}{L})} = \cdots = \dfrac{\sqrt{2}}{8} > 0$ at $L=2$.

(3) Easier: minimize v^2. $\dfrac{dv^2}{dL}=\dfrac{1}{2}-\dfrac{2}{L^2}=0$ at $L=2$, and $\dfrac{d^2v^2}{dL^2}=\dfrac{4}{L^3}>0$.)

(2) $f'(x)=\dfrac{1}{1+x^2}\cdot 2x$ and $f'(x)=0$ if and only if $x=0$ (critical point). A global extreme must occur at an endpoint ($x=-3,2$) or a critical point ($x=0$): $f(-3)=\ln 10$, $f(0)=\ln 1=0$, and $f(2)=\ln 5$, so the global max occurs at $x=-3$ and the global min at $x=0$.

(3) In the triangle given, let x be the length of the horizontal leg, and let z be the length of the hypotenuse. By the Pythagorean theorem, $x^2+300^2=z^2$. Therefore, $2x\dfrac{dx}{dt}=2z\dfrac{dz}{dt}$. When $z=500$, $x=\sqrt{500^2-300^2}=400$ and $\dfrac{dz}{dt}=20$, so $2\cdot 400\cdot\dfrac{dx}{dt}=2\cdot 500\cdot 20$, whence $\dfrac{dx}{dt}=25$ ft/sec.

(4) (a) Use $f(x)\approx f(a)+f'(a)(x-a)$ with $f(x)=x^{1/4}, a=1$: $x^{1/4}\approx 1^{1/4}+\dfrac{1}{4}(1)^{-3/4}(x-1) = 1+\dfrac{1}{4}(x-1)$. (b) $(1.1)^{1/4}\approx 1+\dfrac{1}{4}(.1)=1.025$. (c) $\dfrac{d^2}{dx^2}x^{1/4}=-\dfrac{3}{16}x^{-7/4}<0$, so the graph is concave down and the tangent line approximation is an **overestimate**.

(5) (a) $\displaystyle\lim_{x\to 0}\frac{6^x-2^x}{x} = \lim_{x\to 0}\frac{(\ln 6)6^x-(\ln 2)2^x}{1} = \ln 6-\ln 2=\ln 3$, by l'Hospital's rule for form 0/0.

(b) $\displaystyle\lim_{x\to\pi}\frac{\sin(2x)}{1-\cos(2x)} = \frac{\sin(2\pi)}{1-2\cos(\pi)} = \frac{0}{3}=0.$

(c) $\displaystyle\lim_{x\to 0}\frac{e^x-x-1}{x^2} = \lim_{x\to 0}\frac{e^x-1}{2x} = \lim_{x\to 0}\frac{e^x}{2}=\frac{1}{2}$, by l'Hospital's rule twice.

(6) Length $L=8x+3y$ is to be minimized. The area $(x+3x)\cdot y=9600$, so $y=2400/x$. Therefore, $L=8x+3(2400/x)=8x+7200/x$ for $0<x<\infty$. Now $\dfrac{dL}{dx}=8-\dfrac{7200}{x^2}$, and $\dfrac{dL}{dx}=0\Leftrightarrow 8x^2=7200\Leftrightarrow x^2=900\Leftrightarrow x=30$, since $x>0$. Here $\dfrac{d^2L}{dx^2}=\dfrac{14400}{x^3}>0$, so L is minimized where $\dfrac{dL}{dx}=0$, i.e, $x=30$ feet and $y=80$ feet.

(*Alternative tests*: (1) Check "endpoints": $L\to\infty$ as $x\to 0^+$ and as $x\to\infty$; (2) $0<x<30\Rightarrow dL/dx=8-7200/x^2<0$ and $x>30\Rightarrow dL/dx=8-7200/x^2>0$.)

(7) (a) $f'(x)=12x^2-4x^3=4x^2(3-x)$, so the critical points are $x=0$ and $x=3$. If $x<0, f'(x)>0$; if $0<x<3, f'(x)>0$; if $x>3, f'(x)<0$. Therefore f is increasing on $(-\infty,3]$, and f is decreasing on $[3,\infty)$.

(b) (a) also shows that $f(3)=27$ is a local maximum, and there is no local minimum.

(c) $f''(x)=24-12x^2=12x(2-x)$, and $f''(x)$ changes sign from $-$ to $+$ to $-$ at $x=0$ and $x=2$, so these (or $(0,0)$ and $(2,16)$) are the inflection points, and f is concave up on $[0,2]$ and concave down on $(-\infty,0]$ and on $[2,\infty)$.

(8) (a) Let $q =$ the number of passengers; let $p =$ the price. Then

$$p = \begin{cases} 2000 & \text{if } 0 \le q \le 100 \\ 2000 - 10(q - 100) = 3000 - 10q & \text{if } 100 < q \le 200, \end{cases}$$

so revenue

$$R = pq = \begin{cases} 2000q & \text{if } 0 \le q \le 100 \\ 3000q - 10q^2 & \text{if } 100 < q \le 200. \end{cases}$$

Therefore, marginal revenue

$$\frac{dR}{dq} = \begin{cases} 2000 & \text{if } 0 \le q < 100 \\ \text{undefined} & \text{if } q = 100 \\ 3000 - 20q & \text{if } 0 < q \le 200, \end{cases}$$

and $\frac{dR}{dq} = 0$ if and only if $3000 - 20q = 0$, i.e., $q = 150$. Checking the endpoints (0 and 200) and critical points (100 and 150) we find

q	0	100	150	200
R	\$0	\$200,000	\$350,000	\$200,000

Thus, revenue is maximized when Trump has 150 passengers, and the price is then \$1500.

(b) Total cost $C = 100,000 + 500q$ $(0 \le q \le 200)$.