MCS 121–Spring 2005: Evening Exam 2 answers

(1) (a) $f'(x) = 20x^4 + 20x^3 = 20x^3(x+1)$.
The critical points are where $f'(x) = 0$: at $x = 0$ and $x = -1$.
Since $f'(x) < 0$ only for $-1 < x < 0$, f is decreasing on $[-1, 0]$, and f is increasing on $(-\infty, -1]$ and $[0, \infty)$.

(b) Because $f'(x)$ changes sign from $+$ to $-$ at $x = -1$, $f(x)$ has a local maximum there, and because $f'(x)$ changes sign from $-$ to $+$ at $x = 0$, $f(x)$ has a local minimum there

(c) $f''(x) = 80x^3 + 60x^2 = 80x^2(x + 3/4)$. Thus the only sign change (from $-$ to $+$) for $f''(x)$ occurs at $x = -3/4$, the only inflection point. Also, f is concave down for $x < -3/4$ and concave up for $x > -3/4$ (or $-3/4 < x < 0$ and $0 < x < \infty$).

(2) $f(x) = e^{1+x^2}$ for $-1 \le x \le 2$, and so $f'(x) = e^{1+x^2} \cdot 2x$ there. Thus, $f'(x) = 0$ iff $x = 0$, the critical point. The endpoints are $x = -1$ and $x = 2$.

x	-1	0	2
$f(x)$	e^2	e	e^5

Thus $f(x)$ has its global minimum at $x = 0$ and its global maximum at $x = 2$.

(3) (a) $f(x) = x^{-1/2}$, so $f'(x) = (-1/2)x^{-3/2} = \dfrac{-1}{2\sqrt{x^3}}$ and $f(x) \approx f(4) + f'(4)(x-4) = \dfrac{1}{2} + \dfrac{-1}{2\sqrt{4^3}}(x-4) = \dfrac{1}{2} - \dfrac{1}{16}(x-4)$, or $f(x) \approx 3/4 - x/16$.

(b) $f''(x) = \dfrac{3}{4}x^{-5/2} > 0$, so f is concave up and the tangent line approximation yields an underestimate.

(c) Let $x = 5$ in (a): $\dfrac{1}{\sqrt{5}} \approx \dfrac{1}{2} - \dfrac{1}{16}(5-4) = \dfrac{7}{16} \overset{\text{or}}{=} 0.4375$.

(4) (a) $\displaystyle\lim_{x\to 3} \frac{x^2 - 4x + 3}{x^2 + x - 12} = \lim_{x\to 3} \frac{2x-4}{2x+1} = \frac{2}{7}$, by L'Hospital's rule.

(b) $\displaystyle\lim_{x\to 0} \frac{1+e^x}{e^{1+x}} = \frac{1+e^0}{e^{1+0}} = \frac{2}{e}$.

(c) $\displaystyle\lim_{x\to 0} \frac{\cos(x) - 1}{x^2} = \lim_{x\to 0} \frac{-\sin(x)}{2x} = \lim_{x\to 0} \frac{-\cos(x)}{2} = -\frac{1}{2}$, by L'Hospital's rule (twice).

(5) We are given $\dfrac{d\theta}{dt} = 6\pi$ radians per minute, where θ is the angle at the top of the diagram, and we must find $\dfrac{dx}{dt}$ when $\theta = \pi/4$, where x is the distance from P to the lit spot on the shoreline. So, relate θ to x: $\tan(\theta) = x/200$, or $x = 200\tan(\theta)$. Therefore, $\dfrac{dx}{dt} = 200\sec^2(\theta)\dfrac{d\theta}{dt} = \dfrac{200}{\cos^2(\pi/4)} \cdot 6\pi = \dfrac{1200\pi}{(\sqrt{2}/2)^2} = 2400\pi \approx 7540$ meters per minute, or about 126 meters per second. (Cf. §3.6 #54)

(6) Let x and y be the positive numbers, and let $S = x + \ln y$. We have $x + y = 10$, so $y = 10 - x$ and $S = x + \ln(10 - x)$. Now $\dfrac{dS}{dx} = 1 + \dfrac{-1}{10 - x}$, and $\dfrac{dS}{dx} = 0$ iff $10 - x = 1$. Thus $x = 9$ is a critical point. Because dS/dx changes sign from $+$ to $-$ at $x = 9$, or because $d^2S/dx^2 = -(10-x)^{-2} < 0$, we know the maximum value of $S = 9 + \ln 1 = 9$ when $x = 9$ and $y = 1$.

(7) Let x be the side of the square base, and let y be the height (both in inches). The volume is given by $x^2y = 288$, so $y = 288/x^2$. Then the cost C is given for $x > 0$ by

$$C = 3(x^2 + 4xy) + 5x^2 = 3x^2 + 12x \cdot \frac{288}{x^2} + 5x^2 = 8x^2 + \frac{3456}{x}.$$

Therefore, $\dfrac{dC}{dx} = 16x - \dfrac{3456}{x^2}$, so $\dfrac{dC}{dx} = 0$ iff $16x = \dfrac{3456}{x^2}$ iff $x^3 = 216$ iff $x = 6$. Since $\dfrac{d^2C}{dx^2} = 16 + \dfrac{6912}{x^3} > 0$ (or $\dfrac{dC}{dx} = \dfrac{16(x^3 - 216)}{x^2}$ changes sign from $-$ to $+$ at $x = 6$), the cost is minimized when $x = 6''$ and $y = 288/6^2 = 8''$.

(8) (a) Fixed cost TFC $= 127$.

(b) $\pi(q) = R(q) - C(q) = -\dfrac{1}{3}q^3 + 529q - 127$.

(c) $\pi'(q) = -q^2 + 529$ for $q \geq 0$, so $\pi'(q) = 0$ iff $q = \sqrt{529} = 23$. Since $\pi''(q) = -2q < 0$ (or $\pi'(q)$ changes sign from $+$ to $-$ at $q = 23$), $q = 23$ items will maximize the profit.