MCS 122: Probability Distributions

Many random quantities have, at least approximately, a so-called exponential distribution. The exponential density \( p(x) \) is given by

\[
p(x) = \begin{cases} \frac{c e^{-cx}}{0} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}
\]

The “parameter” \( c \) is a positive constant. For example, the (random) time (in minutes) from any given point in time until the time a phone call arrives at a switchboard may have an exponential distribution with \( c = 0.5 \).

1. Show that \( \int_{-\infty}^{\infty} p(x) \, dx = 1. \) Thus, \( p(x) \) is a proper probability density.

\[
\int_{-\infty}^{\infty} p(x) \, dx = \int_{-\infty}^{0} p(x) \, dx + \int_{0}^{\infty} p(x) \, dx
\]

\[
= \lim_{a \to -\infty} \int_{a}^{0} 0 \, dx + \lim_{b \to \infty} \int_{0}^{b} c e^{-cx} \, dx
\]

\[
= 0 + \lim_{b \to \infty} \left. \left(-e^{-cx}\right) \right|_{0}^{b}
\]

\[
= \lim_{b \to \infty} \left(-e^{-cb} \right) - \left(-e^{0}\right) = 1.
\]

2. Calculate the probability that the time until the next phone call arrives at the switchboard is between 2 and 4 minutes from now. (Assume \( c = .5 \).)

\[
\text{Probability} = \int_{2}^{4} 0.5e^{-0.5x} \, dx = -e^{-0.5x} \bigg|_{2}^{4} = -e^{-2} + e^{-1} \approx 0.23.
\]

3. Calculate the cumulative distribution function \( P(t) = \int_{-\infty}^{t} p(x) \, dx \), for any \( c > 0 \). Take care to consider two cases: \( t < 0 \) and \( t \geq 0 \).

\[
P(t) = \begin{cases} \int_{-\infty}^{t} 0 \, dx = 0 & \text{if } t < 0; \\ \int_{0}^{t} 0 \, dx + \int_{0}^{t} c e^{-cx} \, dx = -e^{-ct} \bigg|_{0}^{t} = -e^{-ct} + 1 & \text{if } t \geq 0. \end{cases}
\]
4. Calculate the median of an exponential distribution. The answer will depend on $c$, of course.

Let $m$ denote the median. Then $P(m) = 1/2$, so

$$1 - e^{-cm} = \frac{1}{2},$$

whence

$$-e^{-cm} = -\frac{1}{2},$$

and

$$-cm = \ln(1/2) = \ln 1 - \ln 2 = -\ln 2,$$

so

$$m = \frac{\ln 2}{c}.$$

5. Calculate the mean of an exponential distribution. (Note: $\lim_{x \to \infty} x/e^{cx} = \lim_{x \to \infty} 1/(ce^{cx}) = 0$, by l’Hôpital’s rule.)

$$\text{Mean} = \int_{-\infty}^{\infty} xp(x) \, dx$$
$$= \int_{0}^{\infty} x \cdot ce^{-cx} \, dx \quad [\text{since } p(x) = 0 \text{ for } x < 0]$$
$$= \lim_{b \to \infty} \left[ x(-e^{-cx}) - \int_{0}^{b} -e^{-cx} \, dx \right]$$
$$= 0 - \lim_{b \to \infty} \frac{-e^{-cx}}{-c} \bigg|_{0}^{b} = \frac{1}{c}.$$

6. For the switchboard example ($c = 0.5$), what are median and mean of the random waiting time until the next call? Sketch the density and the cumulative distribution function for this case, and label median and mean values on the time axes.

See the Maple output for the graphs.

$$\text{Median} = \frac{\ln 2}{0.5} \approx 1.39 \text{ min.}$$

$$\text{Mean} = \frac{1}{0.5} = 2 \text{ min.}$$