MCS 256 DISCRETE MATHEMATICS
Set 6: Binomial Coefficients—Solutions

1. | 11 | 1 | 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 | 11 | 1 |
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>12</td>
<td>1</td>
<td>12</td>
<td>66</td>
<td>220</td>
<td>495</td>
<td>792</td>
<td>924</td>
<td>792</td>
<td>495</td>
<td>220</td>
<td>66</td>
<td>12</td>
</tr>
</tbody>
</table>

2. (a) HHHH HHTT HHTH HTHT HTHH HTHH THTT THHT TTHH TTTH TTHT THTH THHH

There are 16 possible outcomes.

<table>
<thead>
<tr>
<th>$k$</th>
<th>Number with $k$ heads</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4 HHTT THTT TTHT TTTH</td>
</tr>
<tr>
<td>2</td>
<td>6 HHTT HTHT HTHH THHT THTH TTHH</td>
</tr>
<tr>
<td>3</td>
<td>4 HHHT HHHT HTHH THHH</td>
</tr>
<tr>
<td>4</td>
<td>1 HHHH</td>
</tr>
</tbody>
</table>

(c) There are $2^n$ possible outcomes of $n$ coin tosses, if just the sequence of “heads” and “tails” is noted, because there are 2 possibilities for the first toss, 2 possibilities for the second toss, ..., and 2 possibilities for the $n^{th}$ toss, whence there are $2 \times 2 \times \cdots \times 2 = 2^n$ possibilities.

(d) The number of possible sequences of $n$ “H”的s and “T”的s with exactly $k$ “H”的s is $\binom{n}{k}$, because this is the number of choices for the $k$ positions out of 1..$n$ for the “H”的s (with the remaining $n - k$ positions necessarily filled with “T”的s).

ABCD ABDC ACBD ACDB ADCB ADCB BACD BADC

3. (a) BCAD BCDA BDAC BDCA CABD CADB CBAD CBDA CDAB CDBA DABC DABC DBAC DBCA DCAB DCBA

There are 24 linear arrangements of “A,” “B,” “C,” “D.”

<table>
<thead>
<tr>
<th>$k$</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4 $4^0 = 1$</td>
</tr>
<tr>
<td>1</td>
<td>4 AB AC AD BA BC BD</td>
</tr>
<tr>
<td>2</td>
<td>12 CA CB CD DA DB DC</td>
</tr>
<tr>
<td>3</td>
<td>24 ABC ABD ACB ACD ADB ADC BAC BAD</td>
</tr>
<tr>
<td>4</td>
<td>24 BCA BCD BDA BDC CAB CAD CBA CBD</td>
</tr>
</tbody>
</table>

(b)
(c) There are \(n!\) different linear arrangements of \(n\) distinct items, because there are \(n\) choices for the first item, and then there are \(n - 1\) choices for the second item, and then there are \(n - 2\) choices for the third item, ..., and finally \(n - (n - 1) = 1\) choice for the \(n^{th}\) item, so there are \(n \times (n - 1) \times (n - 2) \times \cdots \times 1\) possibilities.

(d) There are \(n^k\) different linear arrangements of \(k\) distinct items selected from \(n\) items, because there are \(n\) choices for the first item, and then there are \(n - 1\) choices for the second item, and then there are \(n - 2\) choices for the third item, ..., and finally \(n - k + 1\) choices for the \(k^{th}\) item, so there are \(n \times (n - 1) \times (n - 2) \times \cdots \times (n - k + 1)\) possibilities.

4. (a) \[
\sum_{0 \leq k \leq n} \binom{n}{k} = \sum_{0 \leq k \leq n} \binom{n}{k} \cdot 1^{n-k} = (1 + 1)^n = 2^n.
\]

(b) \[
\sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} = \sum_{0 \leq k \leq n} \binom{n}{k} (-1)^k 1^{n-k} = (-1 + 1)^n = 0^n = [n = 0].
\]

This sum is 0 unless \(n = 0\); then it is 1.

(c) \[
\sum_{0 \leq k \leq n} \binom{n}{k} = 2^{n-1}, \text{ if } n > 0, \text{ and the sum } = 1 \text{ if } n = 0,
\]

because if you add the first two equations below you get the third.

\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n
\]

\[
\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = [n = 0]
\]

\[
2 \binom{n}{0} + 2 \binom{n}{2} + \cdots + 2 \binom{n}{n} = 2^n + [n = 0].
\]

5. If \(m \geq 0\), then

\[
\sum_{0 \leq k \leq n} \binom{k}{m} = \sum_{0 \leq k \leq n} \frac{k^m}{m!}
\]

\[
= \frac{1}{m!} \sum_{0 \leq k \leq n+1} k^m
\]

\[
= \frac{1}{m!} \left[ \frac{k^{m+1}}{m+1} \right]^{n+1}_0
\]

\[
= \frac{1}{m!} \cdot \frac{(n+1)^{m+1}}{m+1} - 0 = \frac{(n+1)^{m+1}}{(m+1)!} = \binom{n+1}{m+1}.
\]

This proves the upper summation identity. If \(m < 0\), then the given sum is zero, because all of its terms are 0.