

## Exercises in Asymptotics—Answers

1. Which function/sequence grows faster than the other? [GKP 489]

(a)  $n^{\ln n} \prec (\ln n)^n$

(b)  $n^{\ln \ln \ln n} \prec (\ln n)!$

2. Let  $f(n) = \sum_{k=1}^n \sqrt{k}$ .

(a) Show that  $f(n) = \Theta(n^{3/2})$ , i.e.,  $f(n) \asymp n^{3/2}$ .

(b) Find a function/sequence  $g(n)$  for which  $f(n) = g(n) + O(\sqrt{n})$  and  $f(n) \sim g(n)$ .  
Give support for your answer, of course.

Comparing areas we can deduce

$$\frac{2}{3}n^{3/2} = \int_0^n \sqrt{x} dx < \sum_{k=1}^n \sqrt{k} < \int_1^n \sqrt{x} dx + \sqrt{n} = \frac{2}{3}n^{3/2} - \frac{2}{3} + \sqrt{n}.$$

From this we can finish (a) and (b) with  $g(n) = (2/3)n^{3/2}$ .

3. By means of Stirling's approximation, determine what  $\binom{2n}{n}$  is asymptotic to.

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}.$$

4. Recall that the  $n^{\text{th}}$  harmonic number  $H_n$  is given by

$$H_n = \sum_{k=1}^n \frac{1}{k}.$$

Justify as much of the following asymptotic approximation as you can:

$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

*Euler's constant*

$$\gamma := \lim_{n \rightarrow \infty} (H_n - \ln n)$$

is approximately 0.5772. It is a long-standing open problem to determine whether Euler's constant is rational or irrational.

For starters, by comparing the areas of inscribed and circumscribed rectangles of base 1 with areas under the graph of  $y = 1/x$ , we see  $H_n > \ln(n+1)$  and  $H_n - 1 < \ln n$ , so  $\ln(n+1)/n < H_n - \ln n < 1 \dots$