MCS 256 Discrete Calculus and Probability
Counting Problems: Answers

I.: Basic Principles

A.: Subset $S \leftrightarrow 1 + ([a \in S][b \in S][c \in S])_2$.

B.: $26 + 26 + 10 = 62$.

B.3./I.D./II.C.: $32 + 32 + 32 - 1 - 1 - 1 + 0 = 93$.

B.3.: $\sum_{k=1}^{26} k = (26)(27)/2 = 351$.

C.: $26 \times 10 = 260$.

D.: 2^{2^n}.

E.: $\sum_{k=0}^{n} \frac{(-1)^k}{k!} \approx e^{-1}$ where n = number of items permuted.

F.: **1:** $C(n,k) = \frac{n^{\underline{k}}}{k!}$.

2: $C(n,0) = 1, C(0,k) = [k = 0]$, and $C(n,k) = C(n-1,k) + C(n-1,k-1)$ for $n > 0$.

3: $[x^k](1+x)^n$

II.: Linearly Ordered Arrangements

Assume a 26-letter alphabet is used.

A.: $26^{\underline{5}} = 7893600$.

B.: $26^5 = 11881376$.

C.1.: $\frac{10!}{1!1!2!3!3!} = 50400$.

C.2.: #{ AAA,AAB,ABA,BAA,ABB,BAB,BBA,BBC,BCB,CBB, AAC,ACA,CAA,ABC,ACB,BAC,BCA,CAB,CBA } = 19.

Or, $\frac{3!}{3!0!0!} + \frac{3!}{2!1!0!} + \frac{3!}{1!2!0!} + \frac{3!}{0!2!1!} + \frac{3!}{2!0!1!} + \frac{3!}{1!1!1!} = 1+3+3+3+3+6 = 19$.

C.3.: $26^5 - \left[\binom{5}{3}25^2 + \binom{5}{4}25 + 1\right] - \left[\binom{5}{2}25^3 + \binom{5}{3}25^2 + \binom{5}{4}25 + 1\right] - \left[\binom{5}{4}25 + 1\right] + \binom{5}{3} = 11712258$, by inclusion-exclusion.

III.: Unordered Selections

We are at a shop that offers 26 different flavors of ice cream.

A.1: $\binom{26}{5} = 65780$.

A.2: $\binom{n}{k}$

B.1.: $\left(\binom{26}{5}\right) = 142506$.

B.2.: Same as previous problem

C.: 138854 (See class notes.)

IV.: Sampling

(1) $9^4 = 6561$.

(2) $9^{\underline{4}} = 3024$.

(3) $\left(\!\!\binom{9}{4}\!\!\right) = \binom{12}{4} = 495.$

(4) $\binom{9}{4} = 126.$

V.: Distributions/Functions

A.: Distinguishable boxes

A.1.: $3^9 = 19,683.$

A.2: $\left(\!\!\binom{3}{6}\!\!\right) = 28.$

B.: Distinguishable boxes, given numbers in each box

B.1: $\binom{9}{4,3,2} = 1260.$

B.1: (again) $\binom{52}{13,13,13,13} = 53644737765488792839237440000.$

B.?: $1/\left(\!\!\binom{3}{6}\!\!\right) = 1/28.$

C.: Indistinguishable boxes

C.1.a: $\left\{ {8 \atop 3} \right\} = 966.$

$\left\{ {8 \atop 1} \right\} + \left\{ {8 \atop 2} \right\} + \left\{ {8 \atop 3} \right\} = 1094.$

C.1.b: $\omega_8 = 4140$ (Bell number).

C.2: $7, 6+1, 5+2, 5+1+1, 4+3, 4+2+1, 4+1+1+1, 3+3+1, 3+2+2, 3+2+1+1, 3+1+1+1+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1$