Sum of 4th powers

Derivation of a formula for the sum of the fourth powers
of the first n integers
via a divided difference table and the Newton interpolating polynomial

| n | 1^4+...+n^4 | 1st d.d. | 2nd d.d. | 3rd d.d. | 4th d.d. | 5th d.d. | 6th d.d. |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 15/2 | 50/6 | 60/24 = 2.5 | 1/5 | 0 |
| 1 | 1 | 16 | 65/2 | 110/6 | 84/24 = 3.5 | 1/5 | |
| 2 | 17 | 81 | 175/2 | 194/6 | 108/24 = 4.5 | v0 | |
| 3 | 98 | 256 | 369/2 | 302/6 | v1 | | |
| 4 | 354 | 625 | 671/2 | v2 | | | |
| 5 | 979 | 1296 | v3 | | | | |
| 6 | 2275 | v4 | | | | | |
| n | v5 = p_5(n) | | | | | | |

v0 = f[1,2,3,4,5,6] = 1/5 Assume f[2,3,4,5,6,n] = f[1,2,3,4,5,6].

v1 = 4.5 + (n - 2)v0 = 4.1 + n/5

v2 = 302/6 + (n –3)v1 = 1141/30 + (7/2)n + n^2/5

v3 = 672/2 + (n –4)v2 = 5501/30 + (721/30)n + (27/10)n^2 + n^3/5

v4 = 1296 + (n –5)v3 = 2275/6 + (316/5)n + (158/15)n^2 + (17/10)n^3 + n^4/5

v5 = 2275 + (n –6)v4 = (-1/30)n + (1/3)n^3 + (1/2)n^4 + (1/5)n^5

So

1^4+...+n^4 = (-1/30)n + (1/3)n^3 + (1/2)n^4 + (1/5)n^5.

Alternatively, using a row (the top row), we get

f(0) + (n-0)f[0,1] + x(n-1)f[0,1,2] + n(n-1)(x-2)f[0,1,2,3] + n(n-1)(n-2)(n-3)f[0,1,2,3,4]
 + n(n-1)(n-2)(n-3)(n-4)f[0,1,2,3,4,5] + 0
 =
0 + (n)(1) + n(n-1)(15/2) + n(n-1)(n-2)(50/6) + n(n-1)(n-2)(n-3)(2.5)
 + n(n-1)(n-2)(n-3)(n-4)(1/5)
 =
(1/5) n^5 + (½) n^4 + (1/3) n^3 + (-1/30) n