

ASYMPTOTICS

We consider sequences $f : \mathbb{N} \rightarrow \mathbb{R}$ or $f : \mathbb{N} \rightarrow \mathbb{N}$ or sometimes functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}$. Here \mathbb{N} denotes the positive integers, \mathbb{R} the real numbers, and \mathbb{R}^+ the positive real numbers. For asymptotics, we are interested only in the behavior of $f(n)$ for large values of n , so we need only that f be defined for $n > k$ for some k .

ASYMPTOTIC RELATIONS

Often, but not always, we use the following relations in comparing functions $f(n)$ and $g(n)$ that both approach infinity as n approaches infinity.

Here are two equivalent notations that say that $f(n)$ grows more slowly than $g(n)$:

$$f(n) \prec g(n) \text{ if and only if } f(n) = o(g(n)) \text{ if and only if } \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0.$$

The second notation, $o(g(n))$, is Landau's "little oh" notation. For example, if $0 < p < q$ and $b > 1$,

$$\log n \prec n^p \prec n^q \prec b^n.$$

Here is how we denote that $f(n)$ and $g(n)$ have the same rate of growth:

$$f(n) \asymp g(n) \text{ if and only if } |f(n)| \leq C|g(n)| \text{ and } |g(n)| \leq C|f(n)| \\ \text{for some } C \text{ and all } n > \text{some } k.$$

A stronger relation says that " $f(n)$ is asymptotic to $g(n)$ ":

$$f(n) \sim g(n) \text{ if and only if } \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1.$$

For example, $n^3 \asymp n^3$ and $n^3 \sim n^3$

BIG OH, BIG OMEGA, AND BIG THETA

We say $f(n)$ is $O(g(n))$ ("Big Oh of $g(n)$ ") if and only if there are constants C and k such that $|f(n)| \leq C|g(n)|$ for all $n > k$. Think of $O(g(n))$ as a family of functions. Thus,

$$\sum_{j=1}^n j^2 = O(n^3)$$

and

$$\sum_{j=1}^n j^2 = \frac{n^3}{3} + O(n^2)$$

tell us that the sum is in the class of functions eventually bounded by a multiple of n^3 and, more precisely, in the class of functions given by $n^3/3$ plus a function eventually bounded by a multiple of n^2 .

We use Big Omega for lower bounds:

$$f(n) = \Omega(g(n)) \text{ if and only if } |f(n)| \geq C|g(n)| \text{ for some } C > 0.$$

Big Theta specifies an exact order of growth:

$$f(n) = \Theta(g(n)) \text{ if and only if } f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n)).$$

Thus, $f(n) = \Theta(g(n))$ if and only if $f(n) \asymp g(n)$.

STIRLING'S APPROXIMATION

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \text{ (as } n \rightarrow \infty\text{)}.$$

A refined approximation [1, p. 452]:

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + O\left(\frac{1}{n^4}\right)\right).$$

REFERENCES

- [1] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete Mathematics* 2/e, Addison-Wesley, Boston, 1994, chapter 9.
- [2] Kenneth H. Rosen, *Discrete Mathematics and Its Applications* 6/e, McGraw-Hill, Boston, 2007, section 3.2.