Operator Calculus

Operators are functions whose inputs are functions and whose outputs are functions. For example, if the domain is the set of all real-valued functions of a real variable, $\{f|f:\mathbb{R}\to\mathbb{R}\}$, then Δ, E, and I defined by

$$\Delta f(x) := f(x+1) - f(x),$$

$$Ef(x) := f(x+1)$$

$$If(x) := f(x)$$

are operators on this domain. If the domain is restricted to differentiable functions, then

$$Df(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

defines the derivative operator.

Sums, differences, and constant multiples of operators are defined as they are for functions. But the product of operators A and B is defined as composition: $ABf(x) := A(B(f(x))$, and then powers of an operator amount to repeated application of the operator; e.g., $A^3f(x) = A(A(A(f(x))))$.

The operators Δ, E, and I commute—their order of application doesn't matter. It turns out that then **symbolic calculus**, i.e., symbol manipulation, is valid for combinations of these operators.

One application is Newton's advancing difference formula. Since $E = I + \Delta$,

$$E^n = (I+\Delta)^n = \sum_{k=0}^{n} \binom{n}{k} I^{n-k}\Delta^k = \sum_{k=0}^{n} \binom{n}{k} \Delta^k,$$

so

$$f(a+n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(a).$$

For $a = 0$ this says

$$f(n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(0) = \sum_{k=0}^{n} \frac{\Delta^k f(0)}{k!} n^{\underline{k}}.$$

Let

$$P_n(x) := \sum_{k=0}^{n} \binom{x}{k} \Delta^k f(0) = \sum_{k=0}^{n} \frac{\Delta^k f(0)}{k!} x^{\underline{k}}$$

for real x. Then $P_n(x)$ is a polynomial of degree at most n, and it interpolates the function f at $x = 0, 1, \ldots, n$: $P_n(x) = f(x)$ for $x = 0, 1, \ldots, n$. This is the discrete calculus counterpart of the Taylor polynomial

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

This Taylor polynomial of order n (degree $\le n$) for a function f "agrees" with f and its first n derivatives at $a = 0$: $T_n^{(k)}(a) = f^{(k)}(a)$ for $k = 0, 1, \ldots, n$.

Another application is the calculation of higher differences in terms of the values of the function. Since $\Delta = E - I$, we have

$$\Delta^n = (E - I)^n = \sum_{k=0}^{n} \binom{n}{k} E^{n-k}(-I)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} E^{n-k}.$$

Thus, for example,

$$\Delta^3 f(x) = f(x+3) - 3f(x+2) + 3f(x+1) - f(x).$$

Check it out.