Solution of First-Order Linear Recurrence Relations

Given sequences $\langle a_n \rangle$, $\langle b_n \rangle$, and $\langle c_n \rangle$, we shall solve the first-order linear recurrence

$$a_n Y_n = b_n Y_{n-1} + c_n \qquad (n = 1, 2, 3, \ldots)$$

for Y_n, given the initial value Y_0. We'll assume that $a_n \neq 0$ and $b_n \neq 0$.

Choose the "summation factor" s_n so that $s_n b_n = s_{n-1} a_{n-1}$. Multiply the given recurrence by s_n, and let $T_n = s_n a_n Y_n$; we get

$$T_n = T_{n-1} + s_n c_n.$$

The solution of this is clearly

$$T_n = T_0 + \sum_{k=1}^{n} s_k c_k,$$

where $T_0 = s_0 a_0 Y_0 = s_1 b_1 Y_0$, and then $Y_n = T_n/(s_n a_n)$.

By unwinding/unfolding/backtracking the recurrence

$$s_n = \frac{s_{n-1} a_{n-1}}{b_n}.$$

we get a formula for s_n:

$$s_n = s_1 \frac{a_1 a_2 \cdots a_{n-1}}{b_2 b_3 \cdots b_n}.$$

Therefore

$$Y_n = \frac{1}{s_n a_n} \left(s_1 b_1 Y_0 + \sum_{k=1}^{n} s_k c_k \right).$$

In this formula, the s_1's cancel, so we may as well take $s_1 = 1$ and for $n > 1$ use

$$s_n = \frac{a_1 a_2 \cdots a_{n-1}}{b_2 b_3 \cdots b_n}.$$

Example. The average number C_n of comparisons made by quicksort applied to n items in a random order satisfies $nC_n = (n+1)C_{n-1} + 2n$ for $n > 0$ and $C_0 = 0$. This matches our recurrence with $a_n = n, b_n = n+1, c_n = 2n$, and $s_n = \dfrac{1 \cdot 2 \cdot 3 \cdots (n-1)}{3 \cdot 4 \cdot 5 \cdots (n+1)} = \dfrac{2}{n(n+1)}$, so

$$C_n = \frac{1}{\frac{2}{n(n+1)} \cdot n} \left(0 + \sum_{k=1}^{n} \frac{2}{k(k+1)} \cdot 2k \right) = 2(n+1)(H_{n+1} - 1).$$

References

[1] Graham, Knuth, and Patashnik, *Concrete Mathematics: A Foundation for Computer Science* 2/e, Addison-Wesley, Boston, 1994, p. 27.