

Linear Recurrence Relations with Constant Coefficients

Consider the linear recurrence relation with constant coefficients (“LRRwcc”)

$$b_0y_n + b_1y_{n-1} + \cdots + b_dy_{n-d} = f_n$$

for $n \geq d$, or

$$b_0y_{n+d} + b_1y_{n+d-1} + \cdots + b_dy_n = g_n$$

for $n \geq 0$, where b_0, \dots, b_d are given constants with $b_0b_d \neq 0$ and $\langle g_n \rangle = \langle f_{n+d} \rangle$ is a given sequence. If this given sequence is identically zero, the LRRwcc is said to be “homogeneous”; otherwise, “nonhomogeneous.”

The “characteristic polynomial” is $P(x) = b_0x^d + b_1x^{d-1} + \cdots + b_d = \sum_{j=0}^d b_jx^{d-j}$, a polynomial of degree d . In terms of operator calculus with E being the advancing operator ($Ey_n = y_{n+1}$), the LRRwcc may be written

$$P(E)y_n = g_n.$$

Homogeneous Case

KEY IDEA: If $y_n = r^n$, then $P(E)y_n = r^nP(r)$. Thus, if r is a number for which $P(r) = 0$, then $y_n = r^n$ is one solution of the homogeneous LRRwcc $P(E)y_n = 0$.

RECIPE FOR SOLVING A HOMOGENEOUS LRRwcc

1. Solve the “characteristic equation,” $P(r) = 0$.
2. If there are d distinct roots r_1, \dots, r_d , then the general solution of $P(E)y_n = 0$ is given by

$$y_n = C_1r_1^n + \cdots + C_dr_d^n$$

where C_1, \dots, C_d are arbitrary constants. If the initial d values y_0, \dots, y_{d-1} are given, they determine d linear equations for the d constants. (In some cases the roots are not real but are complex numbers. Then, if the original b_0, \dots, b_d and y_0, \dots, y_{d-1} are all real, then the subsequent y_n are actually real numbers, appearances notwithstanding. Using deMoivre’s formula, $(r \cos \theta + i \sin \theta)^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$, it is possible, though tedious, to rewrite the answer in all real terms.)

3. If there are fewer than d distinct roots, then the characteristic polynomial may be written $P(r) = b_0(r - r_1)^{m_1} \cdots (r - r_h)^{m_h}$ where r_1, \dots, r_h are the distinct roots and m_1, \dots, m_h their respective multiplicities. Then the general solution is of the form

$$y_n = (A_1 + A_2n + \cdots + A_{m_1}n^{m_1-1})r_1^n + \cdots + (B_1 + B_2n + \cdots + B_{m_h}n^{m_h-1})r_h^n$$

In other words, the contribution of r_j to the solution is r_j^n times a polynomial in n of degree $m_j - 1$.

Nonhomogeneous Case

This section considers only special sequences $\langle g_n \rangle$. Linear combinations of such sequences may be handled by linearity.

RECIPE FOR SOLVING A NONHOMOGENEOUS LRR_{wcc}

1. Given the nonhomogeneous equation $P(E)y_n = g_n$, first solve the homogeneous equation $P(E)y_n = 0$. Let $y_n^{(h)}$ denote the general homogeneous solution (with the arbitrary constants not yet determined).
2. Try to find a “particular” solution y_n^* of the nonhomogeneous equation as follows.

If $g_n = a^n$,	try	$y_n^* = Aa^n$.
If $g_n = \sin(cn)$ or $\cos(cn)$,	try	$y_n^* = A \sin(cn) + B \cos(cn)$.
If $g_n = a^n \sin(cn)$ or $a^n \cos(cn)$,	try	$y_n^* = a^n(A \sin(cn) + B \cos(cn))$.
If $g_n = n^p$,	try	$y_n^* = A_0 + A_1n + \cdots + A_p n^p$.
If $g_n = n^p a^n$,	try	$y_n^* = a^n(A_0 + A_1n + \cdots + A_p n^p)$.

If this works, let $y_n^{(p)} = y_n^*$ be the particular solution. In the cases involving a^n , this won't work if a is a root of the characteristic equation. In this case, if a is a root of multiplicity m , multiply the first trial solution y_n^* by n^m to get the actual trial solution: $y_n^{(p)} = n^m y_n^*$. Substituting the form for $y_n^{(p)}$ into the nonhomogeneous equation, you'll get an equation that will determine the constants (A and/or B or A_0, \dots, A_p).

3. The general solution is the sum of the general homogeneous solution and the particular nonhomogeneous solution: $y_n = y_n^{(h)} + y_n^{(p)}$. If the initial values y_0, \dots, y_{d-1} are given, then they determine d equations for the d formerly arbitrary constants.