

SOLUTION OF FIRST-ORDER LINEAR RECURRENCE RELATIONS

Given sequences $\langle a_n \rangle$ and $\langle b_n \rangle$, we shall solve the first-order linear recurrence

$$\boxed{y_n = a_n y_{n-1} + b_n \quad (n = 1, 2, 3, \dots)}$$

for y_n , given the initial value y_0 . We'll assume that each $a_n \neq 0$.

Backtracking/unwinding/unfolding, we get

$$\begin{aligned} y_n &= a_n y_{n-1} + b_n \\ &= a_n(a_{n-1}y_{n-2} + b_{n-1}) + b_n &&= a_n a_{n-1} y_{n-2} + a_n b_{n-1} + b_n \\ &= a_n a_{n-1}(a_{n-2}y_{n-3} + b_{n-2}) + a_n b_{n-1} + b_n &&= a_n a_{n-1} a_{n-2} y_{n-3} + a_n a_{n-1} b_{n-2} + a_n b_{n-1} + b_n \\ &= \dots \end{aligned}$$

Thus,

$$\begin{aligned} y_n &= a_n a_{n-1} \cdots a_{n-k} y_{n-k+1} + a_n a_{n-1} \cdots a_{n-k+1} b_{n-k} + \cdots + a_n b_{n-1} + b_n \\ &= \dots \\ &= a_n a_{n-1} \cdots a_1 y_0 + a_n a_{n-1} \cdots a_2 b_1 + a_n a_{n-1} \cdots a_3 b_2 + \cdots + a_n b_{n-1} + b_n. \end{aligned}$$

Let $A_k = a_1 a_2 \cdots a_k$. Then

$$y_n = A_n y_0 + \frac{A_n}{A_1} b_1 + \frac{A_n}{A_2} b_2 + \cdots + \frac{A_n}{A_{n-1}} b_{n-1} + \frac{A_n}{A_n} b_n,$$

or

$$\boxed{y_n = A_n \left(y_0 + \frac{b_1}{A_1} + \frac{b_2}{A_2} + \cdots + \frac{b_n}{A_n} \right) \text{ where } A_k = a_1 a_2 \cdots a_k.}$$

Example: The average number C_n of comparisons made by quicksort applied to n items in a random order satisfies $nC_n = (n+1)C_{n-1} + 2n$ for $n > 0$ and $C_0 = 0$. Thus,

$$C_n = \frac{n+1}{n} C_{n-1} + 2 \text{ for } n > 0.$$

This is a first-order linear recurrence with $a_n = \frac{n+1}{n}$, $b_n = 2$, and $A_n = \prod_{k=1}^n \frac{k+1}{k} = n+1$,

so

$$C_n = A_n \left(C_0 + \sum_{k=1}^n \frac{b_k}{A_k} \right) = (n+1) \left(0 + \sum_{k=1}^n \frac{2}{k+1} \right) = 2(n+1)(H_{n+1} - 1).$$