Special Numbers

Iverson notation

Iverson (the deviser of APL) invented the following handy bracket notation:

$$[\text{statement}] = \begin{cases} 1 & \text{if statement is true;} \\ 0 & \text{if statement is false.} \end{cases}$$

Thus, for example, $[i = j] = 1$ if $i = j$ and $[i = j] = 0$ if $i \neq j$: $[i = j] = \delta_{ij}$, the Kronecker delta.

Binomial Coefficients

Here is the general definition of the binomial coefficient $\binom{r}{k}$:

$\left(\begin{array}{c} r \\ k \end{array} \right) := \frac{r^{\underline{k}}}{k!}$ for real r and integer $k \geq 0$; it is defined to be zero for real r and integer $k < 0$.

For integer $k \geq 0$,

$$\left(\begin{array}{c} r \\ k \end{array} \right) = [x^k](1+x)^r := \text{coefficient of } x^k \text{ in the expansion of } (1+x)^r.$$

For integers k and n with $0 \leq k \leq n$, $\binom{n}{k}$, read "n choose k," counts the number of ways to select a subset of k objects from a set of n objects.

Stirling Numbers of the Second Kind

For integers k and $n \geq 0$,

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \text{coefficient of } x^{\underline{k}} \text{ in } x^n \text{ written as a factorial polynomial.}$$

Thus, $\left\{ \begin{array}{c} n \\ k \end{array} \right\} = 0$ for integer $k < 0$ as well as for integer $k > n$, and for integer $n \geq 0$,

$$x^n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^{\underline{k}} = \sum_{k \in \mathbb{Z}} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^{\underline{k}}.$$

Stirling numbers of the second kind satisfy the recurrence relation

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} + \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\}$$

for integer $n > 0$ and integer k with boundary conditions $\left\{ \begin{array}{c} n \\ 0 \end{array} \right\} = [n = 0]$ and $\left\{ \begin{array}{c} n \\ n \end{array} \right\} = 1$.

The number $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$, read "$n$ subset k," counts the number of partitions of a set of n elements into k nonempty subsets.

Other notations are used for Stirling numbers of the second kind. The classic handbook *AMS 55* [1] uses a notation like $\mathbb{S}_n^{(k)}$ but with a fancy calligraphic "S." Spiegel [3, p. 7] uses the notation S_k^n, while *Combinatorics* [4, p. 37] uses the notation $S(n, k)$.

(Signless) Stirling Numbers of the First Kind

For integers k and $n \geq 0$,

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = [x^k]x^{\overline{n}} = (-1)^{n-k}[x^k]x^{\underline{n}}.$$

Thus, $\left[\begin{matrix} n \\ k \end{matrix} \right] = 0$ for integer $k < 0$ as well as for integer $k > n$, and for integer $n \geq 0$, factorial powers may be expressed in terms of ordinary powers as follows:

$$x^{\overline{n}} = \sum_{k \in \mathbb{Z}} \left[\begin{matrix} n \\ k \end{matrix} \right] x^k \text{ and } x^{\underline{n}} = \sum_{k \in \mathbb{Z}} \left[\begin{matrix} n \\ k \end{matrix} \right] (-1)^{n-k} x^k.$$

Stirling numbers of the first kind satisfy the recurrence relation

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = (n-1) \left[\begin{matrix} n-1 \\ k \end{matrix} \right] + \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right]$$

for integer $n > 0$ and integer k with boundary conditions $\left[\begin{matrix} n \\ 0 \end{matrix} \right] = [n = 0]$ and $\left[\begin{matrix} n \\ n \end{matrix} \right] = 1$. The number $\left[\begin{matrix} n \\ k \end{matrix} \right]$, read as "$n$ cycle k," counts the number of ways to arrange n objects into k cycles.

Definitions and notations for Stirling numbers of the first kind vary. The classic [1] calls

$$S_n^{(k)} = (-1)^{n-k} \left[\begin{matrix} n \\ k \end{matrix} \right]$$

a Stirling number of the first kind. The classic Stirling number is a signed number, but Graham, Knuth, and Patashnik's $\left[\begin{matrix} n \\ k \end{matrix} \right]$ is an unsigned (signless) Stirling number of the first kind, denoted $s'(n, k)$ in *Combinatorics* [4, p. 34]. The latter text uses $s(n, k)$ and Spiegel [3, p. 7] uses s_k^n for signed Stirling numbers of the first kind.

See [2, section 6.1], [1, section 24.1.3], [3, pp. 6-7,20-21], and [4, pp. 33-42] for further information concerning Stirling numbers.

References

[1] Abramowitz and Stegun, eds., *Handbook of Mathematical Functions*, AMS (Applied Mathematics Series) 55, National Bureau of Standards, 1964.

[2] Graham, Knuth, and Patashnik, *Concrete Mathematics: A Foundation for Computer Science* 2/e, Addison-Wesley, Boston, 1994.

[3] Spiegel, Murray R., *Schaum's Outline of Theory and Problems of Calculus of Finite Differences and Difference Equations*, McGraw-Hill, New York, 1971.

[4] Balakrishnan, V. K., *Schaum's Outline of Theory and Problems of Combinatorics*, McGraw-Hill, New York, 1995.