“Continuous” Calculus versus Discrete Calculus

Ordinary calculus deals largely with real-valued functions of a real variable.

Discrete calculus deals largely with sequences, i.e., functions of an integer variable. The function values may be integers, or they may be real (or even complex) numbers.

In discrete calculus it is common to use subscript notation rather than function notation. Thus, instead of writing $f(n)$ (“f of n”) one might write f_n (“f sub n”).

Sum and Product Notation: Delimited Forms

The capital Greek letter Σ (sigma) denotes summation. The delimited form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

may be defined more precisely by the following recursive definition:

$$\sum_{k=1}^{1} a_k := a_1 \text{ and, for integer } n > 1,$$

$$\sum_{k=1}^{n} a_k := \left(\sum_{k=1}^{n-1} a_k \right) + a_n.$$

(The notation “:=” is used for a definitional equality.)

The index k is like a dummy variable in integration. Furthermore, other initial and final values may be used. For example,

$$\sum_{j=3}^{5} j^2 = 3^2 + 4^2 + 5^2.$$

The capital Greek letter Π (pi) is used to denote a product. The delimited form

$$\prod_{k=1}^{n} a_k = a_1 \cdot a_2 \cdot a_3 \cdots a_n$$

may be defined more precisely by the following recursive definition:

$$\prod_{k=1}^{1} a_k := a_1 \text{ and, for integer } n > 1,$$

$$\prod_{k=1}^{n} a_k := \left(\prod_{k=1}^{n-1} a_k \right) \cdot a_n.$$

Notice that a “product” of one factor is the factor itself.

Sums and Products: General Sigma and Pi Notations

Let $f(k)$ be a function of the integer k. Let $P(k)$ be a property of the **integer** k. (It is to be understood without noting it explicitly that k is an integer.) For example, $P(k)$ could be the property that "k is an odd positive integer less than n." Then

$$\sum_{P(k)} f(k)$$

denotes the sum of all the $f(k)$ values for which the property $P(k)$ holds true. For example,

$$\sum_{\substack{0<k<10 \\ k \text{ is odd}}} k^2 = 1^2 + 3^2 + 5^2 + 7^2 + 9^2.$$

This is easier to understand than the equivalent delimited form,

$$\sum_{j=1}^{5} (2j-1)^2.$$

We'll often use the following

$$\sum_{a \le x < b} f(x) = \sum_{k=a}^{b-1} f(k)$$

when a and b are integers with $a < b$. If the property $P(k)$ cannot be satisfied, i.e., if the set $\{k | P(k) \text{ is true}\}$ is empty, then $\sum_{P(k)} f(k) = 0$. The idea is that the sum of zero terms (no terms at all) is zero. For example, $\sum_{0 \le k < n} k^2 = 0$ when $n = 0$, because there is *no* integer k satisfying $0 \le k < 0$.

Similarly, the notation

$$\prod_{P(k)} f(k)$$

denotes the product of of all the $f(k)$ values for which the property $P(k)$ is true. For example

$$\prod_{\substack{p \text{ is prime} \\ p<10}} p^2 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2.$$

An "empty product," i.e., a product for which $\{k | P(k) \text{ is true}\} = \emptyset$, is defined to be 1. Thus,

$$\prod_{1 \le k \le n} k = n!$$

even when $n = 0$ $(0! = 1)$, and when $n > 0$,

$$\prod_{1 \le k \le n} k = \prod_{k=1}^{n} k = 1 \cdot 2 \cdot 3 \cdots n.$$