MCS 256 Discrete Calculus and Probability **Name** ______________.
Exam 2: Asymptotics and combinatorics 19 March 2007

Instructions: This is a closed-book examination. You may, however, use one 3"-by-5" note card and a calculator. Make sure you show **how** *you derive your answers unless not required. Calculate the values of numerical answers, giving small and moderate values exactly and giving large values in scientific notation to four significant digits.*

Honor pledge:

On my honor, I pledge that I have not given, received, or tolerated others' use of unauthorized aid in completing this work.

Please sign when you are finished: (signed) ____________________________

1. (15%) Arrange the following functions of n (sequences) via the relations $\prec$ and $\asymp$ and $\sim$ (assuming $n \to \infty$):

$$e^n, H_n, \lg n, \ln n, n^e, n!, \sqrt{2\pi n}(n/e)^n, (n/e)^n.$$

(You do not have to justify your answer.)

2. (15%)

(a) Use L'Hospital's rule to show that $\lim_{x\to\infty} \dfrac{\ln x}{x} = 0$.

(b) Note that $\ln n^{1/n} = (1/n)\ln n$ and hence deduce that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

(c) Suppose that $f(n) = n^3 + n^{2-1/n}$. Determine whether each of the following asymptotic statements is true (T), or false (F). (No support is required.)

i. $f(n) = n^3 + O(n^2)$.

ii. $f(n) = n^3 + \Theta(n^2)$.

iii. $f(n) = n^3 + o(n^2)$.

3. (8%) How many different passwords are possible if the allowed characters are letters—upper case and lower case—and digits, and the first character must be a letter, and the password must be six, seven, or eight characters long?

4. (8%) How many different license plate "numbers" are there in which one, two, or three letters are followed by one, two, or three digits?

5. (15%) A standard deck of playing cards has 52 cards. Each card is distinguished by its suit and rank. The four suits are clubs, diamonds, hearts, and spades. The thirteen ranks are A (ace), 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), and K (king). Thus, there are $4 \times 13 = 52$ different cards.

   (a) How many different five-card hands may be selected from the 52 cards?

   (b) What is the probability that a random five-card hands is a full house, i.e., has three cards of one rank and two cards of another rank?

6. (8%) Morse code encodes symbols by means of dots ($\cdot$) and dashes ($-$). For example, in International Morse Code, $A = \cdot -$, $N = - \cdot$, and $V = \cdot \cdot \cdot -$. How many different symbols can be encoded using any number from one to four (inclusive) of dots and dashes?

7. (10%) How many nonnegative integer solutions (x_1, x_2, x_3) of the *inequality*

$$x_1 + x_2 + x_3 \leq 6$$

are there?

8. (21%) Suppose that you have six different books.

   (a) How many different ways can all the books be distributed to three people?

   (b) How many of these distributions give one book to the first person, two books to the second person, and three books to the third person?

   (c) How many different ways can the books be divided into three piles?