

*Instructions: This is a closed-book examination. You may, however, use one new handwritten page of notes and a calculator. You will be graded on **how** you derive or support your answers (except in short-answer problems), as well as the actual answers. Take care not to waste your time answering questions that have not been asked!*

(1) (20%) What is the generating function of each of the following sequences? Give simplified answers.

(a) $\langle 2, 2, 2, 2, 2, \dots, 2, \dots \rangle$

(b) $\langle 2^0, 2^1, 2^2, 2^3, \dots, 2^n, \dots \rangle$

(c) $\langle 1, 4, 6, 4, 1, 0, 0, \dots, 0, \dots \rangle$

(d) $\langle h_n \rangle$ where $h_n = \sum_{k=0}^n \binom{r}{k} \binom{s}{n-k}$ and r and s are fixed numbers.

(2) (5%) What sequence $\langle a_k \rangle$ has the ordinary generating function $A(z) = e^z$?

(3) (20%) Give a formula for $\langle a_r \rangle$ as a coefficient of z^r in an appropriate generating function in each of the following cases. Also write each generating function in a simple closed form formula.

(a) a_r is the number of ways to select r jelly beans from a practically unlimited supply of red, green, yellow, and black jelly beans assuming that you have to take at least one of each kind.

(b) a_r is the number of ways to distribute r people into five different rooms assuming that you have to put at least one person into each room.

(4) (10%) Write a formula with undetermined coefficient(s) for the general solution $\langle u_n \rangle$ of the recurrence

$$u_n = 6u_{n-1} - 9u_{n-2}.$$

(5) (25%) Consider the following recurrence relation.

$$g_0 = 9;$$

$$g_1 = 22;$$

$$g_n = g_{n-1} + 6g_{n-2} \quad \text{for } n > 1.$$

Let

$$G(z) = \sum_{n \geq 0} g_n z^n.$$

- (a) Use the recurrence relation to determine an algebraic equation for $G(z)$.
- (b) Solve this equation for $G(z)$. You should get a rational function of z .

- (6) (20%) A classic paper by the Nobel Prize-winning economist Paul Samuelson studied a second-order recurrence model for national income. Let Y_t denote national income (gross domestic product) in period t (which may be a year or a quarter or some other period). Then

$$Y_t = \alpha(1 + \beta)Y_{t-1} - \alpha\beta Y_{t-2} + G$$

gives the national income for $t = 2, 3, 4, \dots$ if Y_1 and Y_0 are given. Here G is the government spending in each time period, assumed constant, α is the marginal propensity to consume, and β is the “relation.” Assume that

$$\alpha = 0.8,$$

$$\beta = 0.25,$$

$$G = 2.$$

Here the government spending G in one time period is measured in trillions of dollars.

- (a) Find a closed-form formula for Y_t . (Your formula should have a couple of constants that in principle could be determined from the initial conditions, but since the initial conditions are not given, the constants cannot be determined.)
- (b) What will the national income be approximately for large t —or exactly for $t \rightarrow \infty$?

- (7) (Extra credit) Do all the regular problems first, because you'll earn points at a slower rate on these problems. Complete Problem 5 as follows.
- (a) Find the partial fractions decomposition of $G(z)$.
 - (b) Determine a closed-form formula for g_n , the solution of the original recurrence relation.