Exercises in Asymptotics

1. Which function/sequence grows faster than the other? [GKP 489]
   (a) $n^{\ln n}$ or $(\ln n)^n$
   (b) $n^{\ln \ln \ln n}$ or $(\ln n)!$
2. Let $f(n) = \sum_{k=1}^{n} \sqrt{k}$.
   (a) Show that $f(n) = \Theta(n^{3/2})$, i.e., $f(n) \asymp n^{3/2}$.
   (b) Find a function/sequence $g(n)$ for which $f(n) = g(n) + O(\sqrt{n})$ and $f(n) \sim g(n)$. Give support for your answer, of course.
3. By means of Stirling's approximation, determine what
   $$\binom{2n}{n}$$
   is asymptotic to.
4. Recall that the n^{th} harmonic number H_n is given by
   $$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$
   Justify as much of the following asymptotic approximation as you can:
   $$H_n = \ln n + \gamma + O(\frac{1}{n}).$$
   Euler's constant
   $$\gamma := \lim_{n\to\infty} (H_n - \ln n)$$
   is approximately 0.5772. It is a long-standing open problem to determine whether Euler's constant is rational or irrational.