A Divide-and-Conquer Recurrence Relation

A divide-and-conquer algorithm that divides a list of n items into equal or approximately parts should divide the list into sublists of $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ items. Suppose that the complexity C_n of such an algorithm satisfies the recurrence relation

$$C_n = C_{\lceil n/2 \rceil} + C_{\lfloor n/2 \rfloor} + n$$

for $n > 1$ with initial condition $C_1 = 0$.

1. Find an exact formula for C_n by carrying out the following steps.
   (a) Let $D_n = C_{n+1} - C_n$. Find a simple recurrence relation for D_n in terms of $D_{\lfloor n/2 \rfloor}$ or $D_{\lceil n/2 \rceil}$. The initial condition is $D_1 = 2$.
   (b) Solve this recurrence for D_n, e.g., by unfolding/unwinding/backtracking.
   (c) Write C_n as a telescoping sum of the D_k terms.
   (d) Tell why the number of bits in the binary representation of k is $\lg \lfloor k \rfloor + 1$.
   (e) What is the number of bits in the binary representations of numbers less than $n = 10$? Look at them.

$$\begin{array}{cccc}
 & & & 1 \\
 & & 1 & 0 \\
 & & 1 & 1 \\
 & 1 & 0 & 0 \\
 & 1 & 0 & 1 \\
 & 1 & 1 & 0 \\
 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{array}$$

   Counting by columns from the right, it is $9 + 8 + 6 + 2$. How are these summands related to $n = 10$? (Hint: Fill out the rectangle with zeros.)
   Generalize. (You should have a sum of $\lg \lfloor n \rfloor + 1$ terms.)
   (f) Thus, determine a closed-form formula for C_n.

2. Assume $n = 2^k$ for some nonnegative integer k. In this case the algorithm for C_n becomes

$$C_n = 2C_{n/2} + n$$

   with $C_1 = 1$.
   (a) Solve this recurrence relation.
   (b) Compare this solution with the general solution obtained in the first part.