MCS 256 Discrete Calculus and Probability **SOLUTIONS**
Exam 5: Final Examination 22 May 2007

Instructions: This is a closed-book examination. You may, however, use one 8.5"-by-11" page of notes, your note cards for previous exams, and a calculator. You may **not** *use the calculator for symbolic computation. Make sure you show* **how** *you derive your answers unless not required. Calculate the values of numerical answers, giving small and moderate values exactly if you can. Otherwise report values to three or four significant digits.*

Honor pledge:

On my honor, I pledge that I have not given, received, or tolerated others' use of unauthorized aid in completing this work.

Please sign when you are finished: (signed) ______________________________

(1) (10%) The number B_n of comparisons done in an unsuccessful binary search of a table of n sorted values in the worst case is determined by the recurrence

$$B_n = B_{\lfloor n/2 \rfloor} + 1$$

for $n \geq 2$ with $B_1 = 1$. Find the exact solution of this recurrence. (A big-oh or big-theta solution is worth partial credit.)
(Aside: Binary search is a recursive algorithm that seeks a target value in a given, sorted list: Look in the middle; if that's not the target, then look (recursively) in the left half or right half according as the target is smaller or larger than the target value.)

Unfold/unwind/backtrack:

$$\begin{aligned} B_n &= 1 + B_{\lfloor n/2 \rfloor} \\ &= 1 + 1 + B_{\lfloor \lfloor n/2 \rfloor /2 \rfloor} = 2 + B_{\lfloor n/4 \rfloor} \\ &= 3 + B_{\lfloor n/2^3 \rfloor} = \cdots \\ &= k + B_{\lfloor n/2^k \rfloor}. \end{aligned}$$

For $k = \lfloor \lg n \rfloor$, $\lfloor n/2^k \rfloor = 1$ and $B_n = k + B_1$, i.e.,

$$B_n = \lfloor \lg n \rfloor + 1.$$

(2) (10%) Let g_k denote the number of ways to give k identical gifts to five different people.

(a) What is the (ordinary) generating function of $\langle g_k \rangle$?

(b) Determine the value of g_9.

(a) $\displaystyle\sum_{k \geq 0} g_k x^k = (1 + x + x^2 + \cdots)^5 = \frac{1}{(1-x)^5}.$

(b) $g_9 = \left(\!\binom{5}{9}\!\right) = \binom{5-1+9}{9} = \binom{13}{4} = 715.$

(3) (10%) The Catalan numbers $\langle C_n \rangle$ satisfy the recurrence

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1} C_0 = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

for $n \geq 1$ with $C_0 = 1$ (and $C_1 = 1$). Let $C(x)$ be the ordinary generating function of $\langle C_n \rangle$. Use the recurrence and initial condition(s) to determine an algebraic equation that $C(x)$ must satisfy. (You are *not* asked to solve it.)

$$\sum_{n\geq 1} C_n x^n = \sum_{n\geq 1} \left(\sum_{k=0}^{n-1} C_k C_{n-1-k} \right) x^{n-1} \cdot x.$$

Therefore,

$$C(x) - C_0 = x \cdot C(x) \cdot C(x).$$

Therefore,

$$x[C(x)]^2 - C(x) + 1 = 0.$$

(This was done in class.)

(4) (15%) Let X be a nonnegative integer-valued random variable. For $n \geq 0$, let $p_n = P(X = n)$. Assume that $p_1 = 1/5$ and $p_n = (1/4)p_{n-1}$ for $n > 1$. Notice that p_0 has not (yet) been specified.

(a) Find a closed-form formula for p_n for $n \geq 1$.

(b) By the postulates of probability, what must be the value of the sum $\sum_{n=0}^{\infty} p_n$?

Thence determine the numerical value of p_0.

(c) What is the probability generating function of X? Give a closed-form formula.

(d) What is the mean, or expected value, of X?

(a) By unwinding, $p_n = \frac{1}{5}\left(\frac{1}{4}\right)^{n-1}$ for $n \geq 1$.

(b) $\sum_{n=0}^{\infty} p_n = 1$ Therefore,

$$p_0 = 1 - \sum_{n=1}^{\infty} \frac{1}{5}\left(\frac{1}{4}\right)^{n-1} = 1 - \frac{1/5}{1 - 1/4} = 1 - \frac{4}{3} \cdot \frac{1}{5} = \frac{11}{15}.$$

(c)

$$\begin{aligned} G_X(z) &= \sum_{n \geq 0} p_n z^n = p_0 + \sum_{n \geq 1} \frac{1}{5}\left(\frac{1}{4}\right)^{n-1} z^n \\ &= \frac{11}{15} + \frac{(1/5)z}{1 - z/4} = \frac{11}{15} + \frac{4z}{20 - 5z} = \frac{44 + z}{60 - 15z}. \end{aligned}$$

(d) $G'_X(z) = \dfrac{(20 - 5z) \cdot 4 - 4z(-5)}{(20 - 5z)^2} = \dfrac{80}{(20 - 5z)^2}$.

$E(X) = G'_X(1) = \dfrac{80}{(20 - 5)^2} = \dfrac{16}{45} = 0.3\overline{5}$.

(5) (15%) Suppose that you have taken out a bank loan of \$10,000. The bank requires that you pay back the loan in equal monthly installments of \$$P$, charging a nominal rate of 9%, i.e., really a rate of $9/12\% = 0.75\%$ each month. Let $B(k)$ be the balance you owe the bank at the end of k months (right after your monthly payment).

(a) Write a recurrence relation for $B(k)$.

(b) Solve the recurrence relation, i.e., give a closed-form formula for $B(k)$ in terms of P.

(c) Determine the amount of the monthly payment you should make in order to pay off the loan in 36 months.

(a) $B(k) = 1.0075B(k-1) - P$ for $k \geq 1$ and $B(0) = 10,000$.

(b) Let $r = 1.0075$. Unwinding the recurrence we get

$$\begin{aligned} B(k) &= rB(k-1) - P \\ &= r[rB(k-2) - P] - P = r^2 B(k-2) - P - rP \\ &= r[r^2 B(k-3) - P - rP] - P = r^3 B(k-3) - P(1 + r + r^2) = \cdots \\ &= r^k B(0) - P(1 + r + r^2 + \cdots + r^{k-1}), \end{aligned}$$

so

$$B(k) = r^k B(0) - P\frac{r^k - 1}{r - 1} = 1.0075^k \cdot 10000 - P\frac{1.0075^k - 1}{.0075}.$$

ALTERNATIVELY, the particular solution $B^{(p)}(k) = p_0$ has $p_0 = 1.0075p_0 - P$, so $p_0 = P/0.0075$. The homogeneous solution is $B^{(h)}(k) = A \cdot 1.0075^k$, so $B(k) = A \cdot 1.0075^k + p_0$. The initial condition forces $A = 10000 - P/0.0075$, so

$$B(k) = (10000 - P/0.0075)1.0075^k + P/0.0075.$$

(c) $B(36) = 0$ implies

$$P = \frac{r-1}{r^{36}-1} r^{36} B(0) = \frac{(0.0075)(1.0075^{36})(10000)}{1.0075^{36} - 1} \doteq \$318.00.$$

(6) (10%) Suppose that a fair coin is tossed $N = 2n$ times.

(a) Give a formula for the probability that the number of heads is equal to the number of tails.

(b) Use Stirling's approximation to deduce an asymptotic approximation for this probability for large n, and calculate the approximate value for $n = 1000$ (i.e., 2000 tosses).

(a) The answer is given by the binomial probability function:

$$P(n \text{ heads and } n \text{ tails}) = \binom{2n}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}.$$

(b) Stirling's approximation says $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Therefore, $\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} =$

$$\frac{(2n)!}{(n!)^2 2^{2n}} \sim \frac{\sqrt{2\pi \cdot 2n}(2n/e)^{2n}}{(\sqrt{2\pi n}(n/e)^n)^2 2^{2n}} = \frac{2\sqrt{\pi n}(2^{2n}n^{2n}/e^{2n})}{2\pi n(n^{2n}/e^{2n})2^{2n}} = \frac{1}{\sqrt{\pi n}}.$$

(This was done in class.)
For $n = 1000$, $1/\sqrt{\pi n} \doteq 0.01784$.

(7) (10%) Arrange the following functions of n (sequences) via the relations $\prec$ and $\asymp$ and $\sim$ (assuming $n \to \infty$). (No supporting work is required.)

$$e^n \qquad H_n \qquad \lg n \qquad \ln n \qquad n^e \qquad n! \qquad n \lg n \qquad n^n$$

$$H_n \sim \ln n \asymp \lg n \prec n \lg n \prec n^e \prec e^n \prec n! \prec n^n.$$

(8) (10%)

(a) What is the definition of $x^{\underline{n}}$, the n^{th} falling factorial power of x when n is a positive integer?

(b) If n, a, and b are positive integers and $a < b$, then $\sum_{a \le x < b} x^{\underline{n}} = ?$

(Give a simplified answer.)

(c) Write k^2 in terms of falling factorial powers of k and thence find a closed-form formula for $\sum_{k=1}^{N} k^2$.

(a) $x^{\underline{n}} = \prod_{j=0}^{n-1}(x-j) = x(x-1)(x-2)\cdots(x-n+1)$.

(b) $\sum_{a \le x < b} x^{\underline{n}} = \left.\frac{x^{\underline{n+1}}}{n+1}\right|_a^b = \frac{b^{\underline{n+1}}}{n+1} - \frac{a^{\underline{n+1}}}{n+1}$.

(c) $k^2 = k(k-1) + k = k^{\underline{2}} + k^{\underline{1}}$, so

$$\sum_{k=1}^{N} k^2 = \sum_{1 \le k < N+1} (k^{\underline{2}} + k^{\underline{1}}) = \left[\frac{k^{\underline{3}}}{3} + \frac{k^{\underline{2}}}{2}\right]_1^{N+1} = \frac{(N+1)^{\underline{3}}}{3} + \frac{(N+1)^{\underline{2}}}{2}, \text{ or}$$

$$\sum_{k=1}^{N} k^2 = \frac{(N+1)(N)(N-1)}{3} + \frac{(N+1)(N)}{2} = (N+1)(N)\left[\frac{N-1}{3} + \frac{1}{2}\right] = \frac{(N+1)(N)(2N+1)}{6}.$$

(9) (10%) The following table gives some of the values of a polynomial function $f(n)$ of degree less than or equal to 3.

n	$f(n)$	1st	2nd	3rd
0	12			
1	17			
2	18			
3	15			
4	8			
5				

(a) Calculate the table of divided differences.
Alternatively, you may instead calculate the first-, second-, and third-order ordinary differences.

(b) Determine the value of $f(5)$

(c) (Extra credit) Determine a formula for $f(n)$.

(a)

n	$f(n)$	1st	2nd	3rd
0	12	5	-2	0
1	17	1	-2	0
2	18	-3	-2	**0**
3	15	-7	$\mathbf{-2}$	
4	8	$\mathbf{-11}$		
5	$\mathbf{-3}$			

(b) $f(5) = -3$.

(c) $v_0 = 0; v_1 = -2 + (n-1)v_0 = -2; v_2 = -3 + (n-2)(-2) = -2n + 1;$
$v_3 = 15 + (n-3)(-2n+1) = -2n^2 + 7n + 12.$

$f(n) = -2n^2 + 7n + 12.$

Have a great summer!