

# MCS 256 Discrete Calculus and Probability Set 1

## Homework rules

- Acknowledge your sources (people and texts).
- In nontrivial problems, show *how* you get your answers.
- Turn in neat, well-written solutions, not messy first drafts. Trim "fringes."
- Do not copy collaborative solutions; write up solutions in your own words.
- Turn in homework on time. Each class day late reduces the possible points by 25%.

"The most useful, and usually the simplest, general proof technique for our work is **mathematical induction**. Mathematical induction is also the principal proof technique in computer science."

—Alan Tucker

## Problems

- (1) **Gambler's ruin.** A gambler repeatedly wagers \$1, winning \$1 with probability  $p$  ( $0 < p < 1$ ) and losing \$1 with probability  $q = 1 - p$ . The gambler keeps betting until s/he goes broke ("ruin") or attains  $\$N$ , where  $N$  is a positive integer. Let  $R_k$  denote the probability that, starting with  $\$k$ , the gambler goes broke. Then

$$R_k = pR_{k+1} + qR_{k-1}$$

for  $0 < k < N$ ,  $R_0 = 1$ , and  $R_N = 0$ . Verify that these conditions are satisfied if, for  $k = 0, 1, 2, \dots, N$ ,

$$R_k = \begin{cases} \frac{(q/p)^N - (q/p)^k}{(q/p)^N - 1} & \text{if } p \neq q \\ 1 - \frac{k}{N} & \text{if } p = q = \frac{1}{2}. \end{cases}$$

- (2) In the Tower of Hanoi set-up, what is the smallest number of moves needed to transfer a tower of  $n$  disks from peg  $A$  to peg  $B$  if direct moves between  $A$  and  $B$  are disallowed. Thus, every move must be to or from the intermediate peg  $C$ , and, by the rules, a larger disk may never lie above a smaller one. [GKP #1.2]
- (3) (a) Find a simple formula for the value of the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}$$

by checking its values for small values of  $n$ .

(b) By mathematical induction, prove that your formula is correct.

- (4) Assume that the sequence  $\langle x_n \rangle$  is given by the recurrence relation

$$x_n = x_{n-1} + n^3 \quad \text{for } n \geq 1$$

and the initial condition

$$x_0 = 0.$$

(a) Unwind (unfold/backtrack) this recurrence.

(b) Calculate the first few values of  $x_n$  and compare them with the triangular numbers.

(c) Conjecture a (correct) closed-form formula for  $x_n$ .

(d) By mathematical induction prove that your formula is correct.

- (5) An *arithmetic sequence* is a sequence of numbers having the property that the difference between consecutive numbers is a constant. Letting  $a$  denote the initial value and  $d$  the common difference, the sequence is

$$a, a + d, a + 2d, a + 3d, \dots, a + (n - 1)d, \dots$$

(a) Find a simple closed-form formula for the sum of the first  $n$  terms of an arithmetic sequence.

(b)  $7 + 10 + 13 + 16 + \dots + 307 = ?$

- (6) A *geometric sequence* is a sequence of numbers having the property that the ratio of consecutive numbers is a constant. Letting  $a$  denote the initial value and  $r$  the common ratio, the sequence is

$$a, ar, ar^2, ar^3, \dots, ar^{n-1}, \dots$$

(a) Prove by mathematical induction that if  $r \neq 1$ , then the sum of the first  $n$  terms of the geometric sequence is given by

$$a \frac{1 - r^n}{1 - r}$$

or, equivalently

$$a \frac{r^n - 1}{r - 1}$$

(b) Determine what the sum is when  $r = 1$ .

(c)  $7 + 7/3 + 7/9 + 7/27 + \dots + 7/3^9 = ?$

It is well worth your while to memorize the geometric sum formula.

- (7) “Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.” [1]

Hints: Use strong induction with four “basis” cases, or use “weak” induction and consider cases (whether or not a 4-cent stamp is used).

#### REFERENCES

- [1] Rosen, Kenneth H., *Discrete Mathematics and Its Applications* 6/e, McGraw Hill, Boston, 2007, p. 287.