

# MCS 256 Discrete Calculus and Probability Set 2

## Homework rules

- Acknowledge your sources (people and texts).
- In nontrivial problems, show *how* you get your answers.
- Turn in neat, well-written solutions, not messy first drafts. Trim "fringes."
- Do not copy collaborative solutions; write up solutions in your own words.
- Turn in homework on time. Each class day late reduces the possible points by 25%.

"Looking back ... the rudiments, at least, of the calculus ought to have been compulsory for me. One ought, after all, to be initiated into the life of the world in which one is going to have to live. I was going to have to live in the Western World ... and the calculus ... is ... one of the characteristic expressions of the modern Western genius."

—Arnold Toynbee

Notation:

$$\Delta f(x) := f(x+1) - f(x) \text{ and } \Delta_h f(x) := \frac{f(x+h) - f(x)}{h}.$$

## Problems

- (1) "Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps." [1]

Hints: Use strong induction with four "basis" cases, or use "weak" induction and consider cases (whether or not a 4-cent stamp is used).

- (2) The present value  $P.V.$  of an annuity that pays  $A$  dollars at the end of each year for the next  $n$  years is given by

$$P.V. = \sum_{k=1}^n \frac{A}{(1+i)^k}$$

where  $i$  is the annual interest rate (expressed decimally).

(a) Find a simple closed-form formula for the present value of an annuity.

(b) Evaluate numerically the present value of an annuity that will pay \$10,000 per year for 20 years if the annual interest rate is 5%

- (3) An annuity *due* makes its first payment immediately. The present value  $P.V.$  of an annuity that pays  $A$  dollars per year starting now and then annually for a total of  $n$  payments is given by

$$P.V. = \sum_{k=0}^{n-1} \frac{A}{(1+i)^k}$$

where  $i$  is the annual interest rate (expressed decimally).

(a) Find a simple closed-form formula for the present value of such an annuity.

(b) You have just won the lottery! You're told that you have won \$120 million, but really what you've won is an annuity that will give you 30 annual payments of \$4

million, starting immediately. Ignoring taxes, and assuming an annual interest rate of 7%, what would be the equivalent “lump sum” that you should get if you elect to get the entire prize now?

- (4) Display the first eleven rows of Pascal’s triangle, i.e., tabulate  $\binom{n}{k}$  for  $0 \leq k \leq n \leq 10$ .
- (5) Derive a formula for  $\Delta \frac{f(x)}{g(x)}$  analogous to the quotient rule for derivatives.
- (6) Using the binomial theorem, derive a formula for  $\Delta_h x^n$  for positive integer  $n$ . Then take the limit as  $h \rightarrow 0$ . Comment on your answer.
- (7) Derive a formula for  $\Delta x^{\bar{n}}$  valid when  $n$  is a positive integer.
- (8) Derive a formula for  $\Delta x^{\underline{n}}$  when  $n = -m$  and  $m$  is a positive integer.

#### REFERENCES

- [1] Rosen, Kenneth H., *Discrete Mathematics and Its Applications* 6/e, McGraw Hill, Boston, 2007, p. 287.