MCS 256 Discrete Calculus and Probability
Set 5: Summation and Asymptotics

Homework rules

- Acknowledge your sources (people and texts).
- In nontrivial problems, show *how* you get your answers.
- Turn in neat, well-written solutions, not messy first drafts. Trim "fringes."
- Do not copy collaborative solutions; write up solutions in your own words.
- Turn in homework on time. Each class day late reduces the possible points by 25%.

Summation Problems

(1) Given sequences $\langle b_n \rangle$ and $\langle c_n \rangle$, solve the first-order linear recurrence

$$y_n = b_n y_{n-1} + c_n \qquad (n = 1, 2, 3, \ldots)$$

for y_n, given the initial value y_0, by backtracking/unwinding/unfolding. Assume that each $b_n \neq 0$.

(2) The purpose of this problem is to show you parallels and contrasts between integration and summation. Evaluate the following definite integrals and sums. Assume that a and b are integers and n is a positive integer. Compile your answers in a table on one page, and attach your readable supporting work on a separate page.

(1) $$\int_0^n x^2 e^x \, dx \qquad \sum_{0 \le k < n} k^2 2^k$$

(2) $$\int_0^n x^3 e^x \, dx \qquad \sum_{0 \le k < n} k^{\underline{3}} 2^k$$

(3) According to [1, p. 599], the average number of comparisons done by a binary search algorithm is given by

$$B(n) = \frac{1}{2n+1} \left[\sum_{r=1}^{k} r 2^{r-1} + k(n+1) \right]$$

for an input list of n items if $n = 2^k - 1$ (and thus $k = \log_2(n+1)$).

(a) Use your knowledge of discrete calculus to derive a simple closed-form formula for $B(n)$ in the given case.

(b) What, approximately, is the average number of comparisons for an input of $2^{20} - 1$ (about a million) items?

(4) (a) Complete the following table.

p	$\sum_{k=0}^{n} k^p$	$\int_0^n x^p\,dx$
1		
2		
3		
4		

(We have already figured out formulas for these sums, either in class or in problem sets, so you need only look up the answers, sometimes perhaps changing an n to an $n+1$.)

(b) Describe the asymptotic relationship (as $n \to \infty$) between the sum and the integral on each line.

(c) Conjecture what the asymptotic relationship is between $\sum_{k=0}^{n} k^p$ and $\int_0^n x^p\,dx$ as $n \to \infty$ for an arbitrary positive integer p.

(5) (A problem by David Wolfe) For each function $f(n)$ and time t in the following table, determine the largest size n of a problem that can be solved in time t, assuming that the algorithm to solve the problem takes $f(n)$ microseconds.
For each row, don't write more than two significant digits. You need only fill the first few entries in some rows: Once you write an entry that is at least 10^{12}, go on to the next row.

$f(n)$	1 second	1 minute	1 hour	1 day	1 month	1 year	1 century
$\lg n$							
$\sqrt{n}$							
n							
$n \lg n$							
n^2							
n^3							
2^n							
$n!$							

(6) (Continuation) For $n = 10^3$, how long a time is $f(n)$ in each case? Answer in seconds or minutes or hours or ... as appropriate. Don't write more than two significant digits. For example, for $f(n) = n \lg n$, the answer is 0.0099 second, i.e., 9.9×10^{-3} second.

References

[1] Alagar, Vangalur S., *Fundamentals of Computing: Theory and Practice*, Prentice Hall, Englewood Cliffs, NJ, 1989.