MCS 341: Normal probability distributions

I'll write $Y \sim N(\mu, \sigma)$ to indicate that the random variable Y has a normal distribution with mean μ and standard deviation σ. Then $N(0, 1)$ denotes the *standard* normal distribution. Its pdf is often denoted ϕ:

$$\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

for $-\infty < y < \infty$.

1. Show that ϕ is indeed a probability density function.

2. Show that if $Z \sim N(0, 1)$, then $E(Z) = 0$ and $V(Z) = 1$. [Hint: Use integration by parts.]

3. Show that if $Z \sim N(0, 1)$, then $Y := \sigma Z + \mu$ has the pdf f where

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right\}$$

   for $-\infty < y < \infty$. Thus, f is indeed a probability density, and $Y := \sigma Z + \mu \sim N(\mu, \sigma)$.

4. Assume only that $E(X) = \mu$ and $V(X) = \sigma^2$ where $\sigma > 0$. (Do not assume at this point that X has a normal distribution or any other specific distribution.) Let

$$Z := \frac{X - \mu}{\sigma}.$$

   The random variable Z is the *standardized* r.v. corresponding to X. Calculate $E(Z)$ and $V(Z)$.

5. Assume that $X \sim N(\mu, \sigma)$. Let $Z := \dfrac{X - \mu}{\sigma}$. Show that Z has a standard normal distribution.