

# FRACTAL DIMENSION OF ARITHMETICAL STRUCTURES OF GENERALIZED BINOMIAL COEFFICIENTS MODULO A PRIME

JOHN M. HOLTE  
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE  
GUSTAVUS ADOLPHUS COLLEGE, ST. PETER, MN 56082

ABSTRACT. Given a sequence  $(u_n)$  of positive integers generated by  $u_1 = 1, u_2 = a, u_n = au_{n-1} + bu_{n-2} (n \geq 3)$ , define the generalized factorial by  $[n]! = u_1 u_2 \cdots u_n$  and the generalized binomial coefficient by  $C(i, j) = [i+j]! / ([i]![j]!)$ . Assume that the prime  $p$  does not divide  $b$ . Let  $r = \min\{n : p | u_n\}$ . **Theorem 1 (Asymptotic abundance of residues):**  $\#\{(i, j) | 0 \leq i, j < rp^k \text{ and } C(i, j) \equiv \rho \pmod{p}\} \sim \frac{r(r+1)}{2(p-1)} \binom{p+1}{2}^k$  as  $k \rightarrow \infty$  for  $\rho = 1, \dots, p-1$ . **Theorem 2 (Fractal dimension):** Let  $s_k = rp^k$ . The Hausdorff dimension of  $\bigcap_k \bigcup_{i, j < s_k} \{[i/s_k, (i+1)/s_k] \times [j/s_k, (j+1)/s_k] : p \nmid C(i, j)\}$  is  $\log \binom{p+1}{2} / \log p$ .

## 1. INTRODUCTION

A classical theorem of E. Lucas [15] expresses the binomial coefficient  $\binom{N}{m}$  modulo a prime  $p$  in terms of the binomial coefficients of the base- $p$  digits of  $N$  and  $m$ : If  $N = \sum N_j p^j$  and  $m = \sum m_j p^j$  where  $0 \leq N_j, m_j < p$ , then

$$\binom{N}{m} \equiv \prod \binom{N_j}{m_j} \pmod{p}.$$

Alternatively, letting

$$B(m, n) := \binom{m+n}{m} = \frac{(m+n)!}{m!n!},$$

we have

$$B(m, n) \equiv B(m \div p, n \div p) B(m \bmod p, n \bmod p) \pmod{p}$$

where  $m \div p$  is the integer quotient of  $m$  by  $p$ , and  $m \bmod p$  is the remainder. As noted in [18], this implies that, modulo  $p$ , the matrix  $[B(m, n) \bmod p]$  with  $0 \leq m, n < p^k$  is equivalent to  $\mathbf{B}^{\otimes k}$ , the  $k$ -fold tensor (or, Kronecker) product of  $\mathbf{B} = [B(i, j) \bmod p]$  where  $0 \leq i, j < p$ . Note that matrix indices start at index pair  $(0, 0)$ . This is an algebraic and “square” representation of the oft-noted self-similarity structure of Pascal’s “triangle”; see, e.g., [19], [2], [7], [8], [14], [22], and [1]. For example, if  $p = 3$ ,

then the matrix  $[B(m, n) \bmod p]$  for  $0 \leq m, n < 9$  is given as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1\mathbf{B} & 1\mathbf{B} & 1\mathbf{B} \\ 1\mathbf{B} & 2\mathbf{B} & 0\mathbf{B} \\ 1\mathbf{B} & 0\mathbf{B} & 0\mathbf{B} \end{bmatrix} \equiv \mathbf{B} \otimes \mathbf{B} \pmod{p},$$

where

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \pmod{p}.$$

The nonzero residues of the matrix  $\mathbf{B}^{\otimes k}$  may be associated with the subset  $B_k$  of  $[0, 1) \times [0, 1)$  formed by taking the union of those squares  $[m/p^k, (m+1)/p^k) \times [n/p^k, (n+1)/p^k)$  for which  $p \nmid B(m, n)$  ( $0 \leq m, n < p^k$ ). Then  $B := \bigcap B_k$  is the union of  $N = p + (p-1) + \cdots + 1 = \binom{p+1}{2}$  self-similar sets. Its “self-similarity dimension” (see Mandelbrot [16], [17, p. 37]), also called the “box-counting dimension” [4, p. 20], is  $D = \log N / \log(1/r)$  where  $r = 1/p$  is the scaling ratio. This result was noted by Wolfram [22] in 1984. Using a different geometric construction, Flath and Peele [5] solved the more difficult problem of determining that the Hausdorff dimension of  $B$  is also  $\log \binom{p+1}{2} / \log p$ . The *Hausdorff dimension*  $\dim_H(B)$  of a subset  $B$  of  $\mathbb{R}^2$  is defined as follows. See, e.g., [4, p. 22]. First, for  $s \geq 0$ , define the Hausdorff measure

$$\mathcal{H}^s(B) = \sup_{\delta > 0} \inf_{\{U_i\}} \sum |U_i|^s$$

where  $|U_i|$  is the diameter of  $U_i$  and the infimum is taken over all countable covers  $\{U_i\}$  of  $B$  with every  $|U_i| \leq \delta$ . Then

$$\dim_H(B) = \inf\{s : \mathcal{H}^s(B) = 0\} = \sup\{s : \mathcal{H}^s(B) = \infty\}.$$

The purpose of this paper is to provide proofs of similar fractal dimension results and density results (previously announced in [11]) for a large class of generalized binomial coefficients. The matrix of generalized binomial coefficients modulo a prime turns out to be formed of basic building blocks arrayed in a pattern that results from superimposing binomial self-similarity upon a doubly periodic “tiling.” The proof relates the enumeration of these building blocks to a Markov chain, and invokes Perron-Frobenius theory to obtain the box-counting-type fractal dimension result. The more challenging Hausdorff dimension result is achieved by employing the mass distribution principle of fractal geometry. Multifractal results have been published elsewhere [9].

## 2. GENERALIZED BINOMIAL COEFFICIENTS

*Generalized binomial coefficients* corresponding to a given sequence  $(u_n)$  are defined analogously to  $B(m, n)$  by replacing  $n!$  by the product of  $u_1$  through  $u_n$ ,

$$[n]! := \prod_{j=1}^n u_j,$$

and then defining

$$C(m, n) := \frac{[m+n]!}{[m]![n]}$$

(assuming any zero factors in the numerator and denominator are first paired and then cancelled).

In this paper we assume that the sequence is defined by a second-order recurrence relation as follows:

$$u_0 = 0; u_1 = 1; u_n = au_{n-1} + bu_{n-2} \text{ for } n = 2, 3, 4, \dots$$

where  $a$  and  $b$  are integers.

When  $a = 2$  and  $b = -1$ , then  $u_n = n$  and the generalized binomial coefficients become the ordinary binomial coefficients:  $C(m, n) = B(m, n)$ . When  $a = 1 + q$  and  $b = -q$ , then  $u_n = 1 + q + q^2 + \dots + q^{n-1}$  and the generalized binomial coefficients are the Gauss  $q$ -binomial coefficients. When  $a = 1$  and  $b = 1$ , then  $u_n = F_n$ , the  $n^{\text{th}}$  Fibonacci number, and the generalized binomial coefficients become the fibonomial coefficients.

## 3. WELLS'S THEOREM AND THE PATTERN OF THE RESIDUES

Wells [20] [21] has proved a generalization of the Lucas theorem for these generalized binomial coefficients. For the purposes of our fractal dimension calculations, we use one of the alternative versions given in [10]. To state it, we need to introduce the following definitions and notations.

**Definition 1.** Let  $r$  denote the rank of apparition of  $p$ ; thus,  $r := \min\{n \in \mathbb{N} : u_n \equiv 0 \pmod{p}\}$ . Let  $t$  denote the (least) period of  $\langle u_n \pmod{p} \rangle$ , if it exists. Let  $s := t/r$ .

**Notation:** If  $r < \infty$ , then for each nonnegative integer  $n$ , let

$$\begin{aligned} n_0 &:= n \bmod r, \\ n' &:= n \div r, \\ n^* &:= n \bmod t, \\ n'' &:= n^* \div r = n' \bmod s. \end{aligned}$$

**Definition 2.** For  $i, j \geq 0$  and for  $0 \leq k, l < r$ , let  $A_{i,j}(k, l)$  denote the solution of the modulo- $p$  recurrence relation

$$A_{i,j}(k, l) \equiv u_{ir+k+1}A_{i,j}(k, l-1) + bu_{jr+l-1}A_{i,j}(k-1, l)$$

for  $0 \leq k, l < r$  together with the boundary conditions

$$A_{i,j}(k, -1) \equiv 0 \pmod{p} \quad \text{for } 1 \leq k < r$$

and

$$A_{i,j}(-1, l) \equiv 0 \pmod{p} \quad \text{for } 1 \leq l < r$$

and

$$A_{i,j}(0, 0) \equiv 1 \pmod{p}.$$

**Definition 3.** For  $i, j \geq 0$  and  $0 \leq k, l < r$ , define

$$H_{i,j}(k, l) := u_{r+1}^{r ij} A_{i,j}(k, l).$$

As noted in [10],  $H_{i,j} \equiv H_{i \bmod s, j \bmod s} \pmod{p}$ , so  $H_{m', n'}(m_0, n_0) \equiv H_{m'', n''}(m_0, n_0) \pmod{p}$ . Also  $H_{m'', n''}(m_0, n_0) \equiv 0 \pmod{p}$  if  $m_0 + n_0 > r$ .

Here is the generalization of Lucas's theorem from [10] that we shall use.

**Proposition 1.** *If  $p \nmid b$ , then, for  $m, n \geq 0$ ,*

$$C(m, n) \equiv B(m', n') H_{m'', n''}(m_0, n_0) \pmod{p}.$$

This result simplifies nicely when  $s = 1$ . Then  $m'' = n'' = 0$ , and  $H_{0,0}(m_0, n_0) \equiv C(m_0, n_0) \pmod{p}$  for  $0 \leq m_0, n_0 < r$ . Thus, in this case, as in the Pascal triangle case, the pattern of residues exhibits self-similarity upon scaling by  $p$ .

**Corollary.** *If  $p \nmid b$  and  $s = 1$ , then, for  $m, n \geq 0$ ,*

$$C(m, n) \equiv B(m', n') C(m_0, n_0) \pmod{p},$$

or, letting  $\mathbf{B}$  denote the matrix  $[B(i, j)]$  with  $0 \leq i, j < p$  and  $\mathbf{C}^{(k)} = [C(m, n)]$  with  $0 \leq m, n < rp^k$ , we have

$$\mathbf{C}^{(k)} \equiv \mathbf{B}^{\otimes k} \otimes \mathbf{C}^{(0)} \pmod{p}.$$

The following examples are borrowed from [10].

**Example 1:  $q$ -binomial coefficients.** Take  $u_n = \sum_{k=0}^{n-1} q^k$  to obtain the  $q$ -binomial coefficients. For a numerical example, take  $q = 2$  and  $p = 5$ . Then  $u_1 = 1, u_2 = 3, u_3 = 7, u_4 = 15, u_5 = 31, \dots$ , whence  $r = 4$ , and

$$\mathbf{C}^{(0)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 7 & 15 \\ 1 & 7 & 35 & 155 \\ 1 & 15 & 155 & 1395 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \pmod{5},$$

so, for  $k = 0, 1, 2, \dots$ ,

$$\mathbf{C}^{(k+1)} \equiv \mathbf{B} \otimes \mathbf{C}^{(k)} \equiv \begin{bmatrix} 1\mathbf{C}^{(k)} & 1\mathbf{C}^{(k)} & 1\mathbf{C}^{(k)} & 1\mathbf{C}^{(k)} & 1\mathbf{C}^{(k)} \\ 1\mathbf{C}^{(k)} & 2\mathbf{C}^{(k)} & 3\mathbf{C}^{(k)} & 4\mathbf{C}^{(k)} & 0\mathbf{C}^{(k)} \\ 1\mathbf{C}^{(k)} & 3\mathbf{C}^{(k)} & 1\mathbf{C}^{(k)} & 0\mathbf{C}^{(k)} & 0\mathbf{C}^{(k)} \\ 1\mathbf{C}^{(k)} & 4\mathbf{C}^{(k)} & 0\mathbf{C}^{(k)} & 0\mathbf{C}^{(k)} & 0\mathbf{C}^{(k)} \\ 1\mathbf{C}^{(k)} & 0\mathbf{C}^{(k)} & 0\mathbf{C}^{(k)} & 0\mathbf{C}^{(k)} & 0\mathbf{C}^{(k)} \end{bmatrix} \pmod{5}.$$

**Example 2: Fibonomial coefficients modulo  $p$ .** Let  $a = b = 1$  so that  $u_n = F_n$ , and consider the case  $p = 3$ . Then  $r = 4$ ,  $t = 8$ , and  $s = 2$ . By Definition 3,

$$\mathbf{H}_{0,0} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \mathbf{H}_{0,1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}; \mathbf{H}_{1,0} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \mathbf{H}_{1,1} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}.$$

The structure of the matrix of fibonomial coefficients modulo 3, in accordance with Proposition 1, is given in Table 1.

|                     |                     |                     |                     |                     |                     |                     |                     |                     |          |
|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|----------|
| $1\mathbf{H}_{0,0}$ | $1\mathbf{H}_{0,1}$ | $1\mathbf{H}_{0,0}$ | $1\mathbf{H}_{0,1}$ | $1\mathbf{H}_{0,0}$ | $1\mathbf{H}_{0,1}$ | $1\mathbf{H}_{0,0}$ | $1\mathbf{H}_{0,1}$ | $1\mathbf{H}_{0,0}$ | $\cdots$ |
| $1\mathbf{H}_{1,0}$ | $2\mathbf{H}_{1,1}$ | $0\mathbf{H}_{1,0}$ | $1\mathbf{H}_{1,1}$ | $2\mathbf{H}_{1,0}$ | $0\mathbf{H}_{1,1}$ | $1\mathbf{H}_{1,0}$ | $2\mathbf{H}_{1,1}$ | $0\mathbf{H}_{1,0}$ | $\cdots$ |
| $1\mathbf{H}_{0,0}$ | $0\mathbf{H}_{0,1}$ | $0\mathbf{H}_{0,0}$ | $1\mathbf{H}_{0,1}$ | $0\mathbf{H}_{0,0}$ | $0\mathbf{H}_{0,1}$ | $1\mathbf{H}_{0,0}$ | $0\mathbf{H}_{0,1}$ | $0\mathbf{H}_{0,0}$ | $\cdots$ |
| $1\mathbf{H}_{1,0}$ | $1\mathbf{H}_{1,1}$ | $1\mathbf{H}_{1,0}$ | $2\mathbf{H}_{1,1}$ | $2\mathbf{H}_{1,0}$ | $2\mathbf{H}_{1,1}$ | $0\mathbf{H}_{1,0}$ | $0\mathbf{H}_{1,1}$ | $0\mathbf{H}_{1,0}$ | $\cdots$ |
| $1\mathbf{H}_{0,0}$ | $2\mathbf{H}_{0,1}$ | $0\mathbf{H}_{0,0}$ | $2\mathbf{H}_{0,1}$ | $1\mathbf{H}_{0,0}$ | $0\mathbf{H}_{0,1}$ | $0\mathbf{H}_{0,0}$ | $0\mathbf{H}_{0,1}$ | $0\mathbf{H}_{0,0}$ | $\cdots$ |
| $1\mathbf{H}_{1,0}$ | $0\mathbf{H}_{1,1}$ | $0\mathbf{H}_{1,0}$ | $2\mathbf{H}_{1,1}$ | $0\mathbf{H}_{1,0}$ | $0\mathbf{H}_{1,1}$ | $0\mathbf{H}_{1,0}$ | $0\mathbf{H}_{1,1}$ | $0\mathbf{H}_{1,0}$ | $\cdots$ |
| $1\mathbf{H}_{0,0}$ | $1\mathbf{H}_{0,1}$ | $1\mathbf{H}_{0,0}$ | $0\mathbf{H}_{0,1}$ | $0\mathbf{H}_{0,0}$ | $0\mathbf{H}_{0,1}$ | $0\mathbf{H}_{0,0}$ | $0\mathbf{H}_{0,1}$ | $0\mathbf{H}_{0,0}$ | $\cdots$ |
| $1\mathbf{H}_{1,0}$ | $2\mathbf{H}_{1,1}$ | $0\mathbf{H}_{1,0}$ | $0\mathbf{H}_{1,1}$ | $0\mathbf{H}_{1,0}$ | $0\mathbf{H}_{1,1}$ | $0\mathbf{H}_{1,0}$ | $0\mathbf{H}_{1,1}$ | $0\mathbf{H}_{1,0}$ | $\cdots$ |
| $1\mathbf{H}_{0,0}$ | $0\mathbf{H}_{0,1}$ | $0\mathbf{H}_{0,0}$ | $0\mathbf{H}_{0,1}$ | $0\mathbf{H}_{0,0}$ | $0\mathbf{H}_{0,1}$ | $0\mathbf{H}_{0,0}$ | $0\mathbf{H}_{0,1}$ | $0\mathbf{H}_{0,0}$ | $\cdots$ |
| $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdots$ |
| $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdots$ |
| $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdot$             | $\cdots$ |

TABLE 1. Submatrices of the fibonomial coefficients mod 3

Proposition 1 and the example show that the infinite matrix  $[C(i, j) \bmod p]$  may be partitioned into  $r \times r$  submatrices which form basic, natural “tiling units.” The pattern of the residues is obtained by superimposing the self-similar array of binomial coefficients modulo  $p$  upon the doubly periodic “tiling” of the plane by “hidden”  $r \times r$   $\mathbf{H}$  matrices. The binomial structure is self-similar upon scaling by the factor  $p$ . The  $r \times r$  tiling structure has period  $s$  both horizontally and vertically, and so the period is  $t$  at the element level. When  $s = 1$ , there are  $p - 1$  different nonzero  $r \times r$  submatrices, one for each nonzero residue value of  $B(m', n') \bmod p$  times  $\mathbf{C}^{(0)}$ . In the general case, there are also  $s \cdot s$  different  $H_{m', n'}$ -matrices. In fact, there are  $(p - 1)s^2$  different nonzero “tiles,” by the following proposition of [10, p. 234].

**Proposition 2.** *Assume  $p \nmid b$ . The number of different nonzero  $r \times r$  submatrices of the infinite matrix  $[C(i, j) \bmod p]$  is  $(p - 1)s^2$ . Furthermore, the mapping  $(\rho, \mu, \nu) \mapsto \rho\mathbf{H}_{\mu, \nu}$  is one to one from  $\{1, \dots, p - 1\} \times \{0, \dots, s - 1\} \times \{0, \dots, s - 1\}$  into the set of  $r \times r$  matrices mod  $p$ .*

In the case of the the fibonomial coefficients modulo 3, the matrix exhibited in Table 1 shows these seven submatrices:

$$1\mathbf{H}_{0,0}, 1\mathbf{H}_{0,1}, 1\mathbf{H}_{1,0}, 1\mathbf{H}_{1,1}, 2\mathbf{H}_{0,1}, 2\mathbf{H}_{1,0}, 2\mathbf{H}_{1,1}.$$

The places of the missing  $2\mathbf{H}_{0,0}$  are farther out—at  $(5, 11), (11, 5), (5, 13), (13, 5) \dots$  in Table 1.

#### 4. SCALING-UP RECURSION FORMULA

Define

$$C_{\alpha,\beta}(m, n) \equiv B(m', n')H_{\alpha+m'', \beta+n''}(m_0, n_0) \pmod{p}.$$

By Proposition 1, if  $p \nmid b$ , then  $C_{0,0}(m, n) \equiv C(m, n) \pmod{p}$ .

**Proposition 3.** *Assume  $p \nmid b$ . If  $m = m_k p^{k-1} r + m^{(k)}$  and  $n = n_k p^{k-1} r + n^{(k)}$  where  $0 \leq m^{(k)}, n^{(k)} < r p^{k-1}$ , then*

$$C_{\alpha,\beta}(m, n) \equiv B(m_k, n_k)C_{\alpha+m_k, \beta+n_k}(m^{(k)}, n^{(k)}) \pmod{p}.$$

*Proof.* Here  $m' := m \div r = m_k p^{k-1} + (m^{(k)})'$ , so, by Lucas's Theorem,  $B(m', n') \equiv B(m_k, n_k)B(m^{(k)'}, n^{(k)'}) \pmod{p}$ . Also  $m'' := m' \bmod s \equiv m_k + m^{(k)'} \pmod{s}$ , because  $p^{k-1} \equiv 1 \pmod{s}$ , a consequence of  $s|p-1$  ([10, p. 229]), so by  $s$ -periodicity,  $H_{\alpha+m'', \beta+n''} \equiv H_{\alpha+m_k+m^{(k)'}, \beta+n_k+n^{(k)'}} \pmod{p}$ . Invoke the definitions of  $C_{\alpha,\beta}(m, n)$  and  $C_{\alpha+m_k, \beta+n_k}(m^{(k)}, n^{(k)})$  to complete the proof.  $\square$

#### 5. ASYMPTOTIC ABUNDANCE OF RESIDUES

Define the matrices

$$\mathbb{C}_{\alpha,\beta}^{(k)} := [C_{\alpha,\beta}(m, n)] \quad (0 \leq m, n < r p^k),$$

and let

$$f_{\alpha,\beta}^{(k)}(\rho, \mu, \nu) := \#\{(i, j) : 0 \leq i, j < p^k, \\ C_{\alpha,\beta}(ir + i_0, jr + j_0) \equiv \rho H_{\mu,\nu}(i_0, j_0) \pmod{p} \text{ for } 0 \leq i_0, j_0 < r\}$$

and

$$f^{(k)}(\rho, \mu, \nu) := f_{0,0}^{(k)}(\rho, \mu, \nu).$$

The quantity  $f^{(k)}(\rho, \mu, \nu)$  is our focus for now. It is the number of  $\rho\mathbb{H}_{\mu\nu}$  tiles in the initial  $r p^k \times r p^k$  square of  $C(i, j)$  values.

**Lemma 1.**

$$f_{\alpha,\beta}^{(k)}(\rho, \mu, \nu) = f^{(k)}(\rho, \mu - \alpha, \nu - \beta).$$

*Proof.* Use the definitions of  $f_{\alpha,\beta}^{(k)}$ ,  $f^{(k)}$ , and  $C_{\alpha,\beta}$ , together with the fact that the mapping  $(\rho, \mu, \nu) \mapsto \rho\mathbf{H}_{\mu,\nu}$  is one to one on  $\{1, \dots, p-1\} \times \{0, \dots, s-1\} \times \{0, \dots, s-1\}$ .  $\square$

**Lemma 2.** For  $1 \leq \rho < p$  and  $0 \leq \mu, \nu < s$ ,

$$f^{(k)}(\rho, \mu, \nu) = \sum_{\substack{0 \leq i, j < p \\ i+j < p}} f^{(k-1)}(B(i, j)^{-1} \rho, \mu - i, \nu - j).$$

Here  $B(i, j)^{-1}$  and  $B(i, j)^{-1} \rho$  are calculated modulo  $p$ .

*Proof.* By Proposition 3,  $\mathbb{C}_{0,0}^{(k)}$  is a  $p \times p$  block matrix whose  $(i, j)$  block is  $B(i, j) \mathbb{C}_{i,j}^{(k-1)}$  ( $0 \leq i, j < p$ ). For  $i + j < p$ , the number of  $\rho H_{\mu, \nu}$  tiles in the  $(i, j)$  block is  $f_{i,j}^{(k-1)}(B(i, j)^{-1} \rho, \mu, \nu)$ . (For  $i + j \geq p$ , we have  $B(i, j) \equiv 0 \pmod{p}$ .) The proof is completed by applying Lemma 1.  $\square$

Let  $\mathcal{J} := \{1, \dots, p-1\} \times \{0, \dots, s-1\} \times \{0, \dots, s-1\}$ . For  $I = (\rho, \mu, \nu)$ ,  $J = (\tilde{\rho}, \tilde{\mu}, \tilde{\nu}) \in \mathcal{J}$ , define

$$\begin{aligned} Q_J^I &:= \#\{(i, j) : 0 \leq i, j < p, i + j < p, B(i, j)^{-1} \tilde{\rho} \equiv \rho \pmod{p}, \\ &\quad \tilde{\mu} - i \equiv \mu \pmod{s}, \quad \tilde{\nu} - j \equiv \nu \pmod{s}\} \\ &= \#\{(i, j) : 0 \leq i, j < p, i + j < p, B(i, j) \rho \equiv \tilde{\rho} \pmod{p}, \\ &\quad \mu + i \equiv \tilde{\mu} \pmod{s}, \nu + j \equiv \tilde{\nu} \pmod{s}\} \end{aligned}$$

and the matrix

$$\mathbb{Q} := [Q_J^I].$$

**Lemma 3.**

$$\sum_{J \in \mathcal{J}} Q_J^I = \frac{p(p+1)}{2} \quad \text{and} \quad \sum_{I \in \mathcal{J}} Q_J^I = \frac{p(p+1)}{2}.$$

*Proof.*

$$\sum_{J \in \mathcal{J}} Q_J^I = \#\{(i, j) : 0 \leq i, j < p, i + j < p\} = \frac{p(p+1)}{2}.$$

Likewise for  $\sum_{I \in \mathcal{J}} Q_J^I$ .  $\square$

Accordingly, let

$$P_J^I = \frac{2}{p(p+1)} Q_J^I.$$

Then

$$\mathbb{P} := [P_J^I]$$

is a doubly stochastic matrix.

**Lemma 4.** Regarding  $f^{(k)}$  as a row vector with indices  $I = (\rho, \mu, \nu)$ , we have

$$f^{(k)} = f^{(0)} \mathbb{Q}^k.$$

*Proof.* By Lemma 2,

$$\begin{aligned} f^{(k)}(\tilde{\rho}, \tilde{\mu}, \tilde{\nu}) &= \sum_{\substack{0 \leq i, j < p \\ i+j < p}} f^{(k-1)}(B(i, j)^{-1} \tilde{\rho}, \tilde{\mu} - i, \tilde{\nu} - j) \\ &= \sum_{(\rho, \mu, \nu) \in \mathcal{J}} f^{(k-1)}(\rho, \mu, \nu) Q_{(\tilde{\rho}, \tilde{\mu}, \tilde{\nu})}^{(\rho, \mu, \nu)} = \sum_{I \in \mathcal{J}} f^{(k-1)}(I) Q_J^I, \end{aligned}$$

so  $f^{(k)} = f^{(k-1)}\mathbb{Q}$ , whence  $f^{(k)} = f^{(0)}\mathbb{Q}^k$ .  $\square$

Note:  $f^{(0)}(\rho, \mu, \nu) = 1$  if  $(\rho, \mu, \nu) = (1, 0, 0)$  and otherwise equals 0.

A nonnegative matrix is *primitive* if some power of it has all positive entries. A Markov chain is *regular* if its transition probability matrix is primitive.

**Lemma 5.** *Every entry of  $\mathbb{Q}^3$  is positive, and so is every entry of  $\mathbb{P}^3$ . Consequently, the finite Markov chain having  $\mathbb{P}$  as its transition matrix is regular.*

*Proof.* Note that  $(\mathbb{Q}^3)_J^I \geq Q_K^I Q_L^K Q_J^L$  and  $Q_{(\rho_2, \mu_2, \nu_2)}^{(\rho_1, \mu_1, \nu_1)} > 0$  if and only if there exists  $(i, j)$  with  $0 \leq i, j < p$  and  $i+j < p$  such that  $B(i, j)\rho_1 \equiv \rho_2 \pmod{p}$ ,  $\mu_1 + i \equiv \mu_2 \pmod{s}$ , and  $\nu_1 + j \equiv \nu_2 \pmod{s}$ . Let  $I = (\rho, \mu, \nu)$  and  $J = (\tilde{\rho}, \tilde{\mu}, \tilde{\nu})$ . For the first factor, let  $i_1 = 1, j_1 = (\rho_1 \rho^{-1} - 1) \pmod{p}, \rho_1 = \tilde{\rho}, \mu_1 = (\mu + i_1) \pmod{s}, \nu_1 = (\nu + j_1) \pmod{s}$ , and  $K = (\rho_1, \mu_1, \nu_1)$ . Then  $Q_K^I > 0$ . Second, let  $i_2 = (\tilde{\mu} - \mu_1) \pmod{s}, j_2 = 0, \rho_2 = \rho_1, \mu_2 = \tilde{\mu}, \nu_2 = \nu_1$ , and  $L = (\rho_2, \mu_2, \nu_2)$ . Then  $Q_L^K > 0$ . Finally, let  $i_3 = 0$  and  $j_3 = (\tilde{\nu} - \nu_2) \pmod{s}$  to show  $Q_J^L > 0$ . Therefore,  $(\mathbb{Q}^3)_J^I > 0$ .  $\square$

The example of the fibonomials modulo 3 shows that 3 is the least power that will work in this lemma.

**Theorem 1.** *For  $1 \leq \rho < p$  and  $0 \leq \mu, \nu < s$ ,*

$$f^{(n)}(\rho, \mu, \nu) \sim \frac{1}{(p-1)s^2} \left[ \frac{p(p+1)}{2} \right]^n \quad \text{as } n \rightarrow \infty$$

and

$$\lim_{n \rightarrow \infty} \frac{\log f^{(n)}(\rho, \mu, \nu)}{\log p^n} = \frac{\log p(p+1)/2}{\log p}.$$

For  $\rho = 0$  we have that

$$\sum_{\mu, \nu} f^{(n)}(\rho, \mu, \nu) = p^{2n} - [p(p+1)/2]^n = p^{2n} [1 - \{(p+1)/(2p)\}^n]$$

is the number of zero tiles in  $\mathbb{C}^{(n)}$

*Proof.* By Lemma 3, the stationary vector of the matrix  $\mathbb{P}$  is  $\frac{1}{(p-1)s^2}(1, \dots, 1)$ . Since  $\mathbb{P}$  is the transition matrix of a regular Markov chain, by Lemma 4, then  $f^{(0)}\mathbb{P}^n$  converges to this stationary vector as  $n \rightarrow \infty$ , by Perron-Frobenius theory (see, e.g., [12, p. 125]). Finally, according to Kummer's theorem [13], the number of pairs  $(i, j)$  with  $0 \leq i, j < p^n$  for which  $p$  does not divide the binomial coefficient  $B(i, j)$  is the same as the number of pairs of  $n$ -digit  $p$ -ary numbers  $(\sum_{k=0}^{n-1} i_k p^k, \sum_{k=0}^{n-1} j_k p^k)$  for which there are no carries when added in base- $p$  arithmetic. This is the same as the number of digit pairs  $(i_k, j_k)$  with  $i_k + j_k < p$ , which is  $[p(p+1)/2]^n$ . Therefore the number of

nonzero tiles  $B(i, j)\rho H_{\mu, \nu}$  in  $\mathbb{C}^{(n)}$  is precisely  $[p(p+1)/2]^n$ , and the number of zero tiles is  $p^{2n} - [p(p+1)/2]^n$ .  $\square$

**Corollary 1.** *Let*

$$R^{(n)}(\rho) := \#\{(i, j) : 0 \leq i, j < rp^n, C(i, j) \equiv \rho \pmod{p}\},$$

*the number of  $C(i, j)$ 's in the initial  $rp^n \times rp^n$  square congruent to  $\rho$  modulo  $p$ . Then the asymptotic abundance of the residue  $\rho$ , where  $1 \leq \rho < p$ , is given by*

$$R^{(n)}(\rho) \sim \frac{r(r+1)}{2(p-1)} \left[ \frac{p(p+1)}{2} \right]^n,$$

*and so the logarithmic density, or box-counting dimension, of the set of generalized binomial coefficients that are congruent to  $\rho$  is*

$$\lim_{n \rightarrow \infty} \frac{\log R^{(n)}(\rho)}{\log p^n} = \frac{\log[p(p+1)/2]}{\log p}.$$

*Proof.* Let

$$g(\rho, \mu, \nu) := \#\{(i, j) : 0 \leq i, j < r, H_{\mu, \nu}(i, j) \equiv \rho \pmod{p}\},$$

the number of entries in the  $r \times r$  matrix  $H_{\mu, \nu}$  that are congruent to  $\rho$  modulo  $p$ . Then

$$R^{(n)}(\rho) = \sum_{1 \leq \tilde{\rho} < p} \sum_{0 \leq \mu, \nu < s} f^{(n)}(\rho \tilde{\rho}^{-1}, \mu, \nu) g(\tilde{\rho}, \mu, \nu),$$

so

$$\begin{aligned} R^{(n)}(\rho) &\sim \sum_{1 \leq \tilde{\rho} < p} \sum_{0 \leq \mu, \nu < s} \frac{[p(p+1)/2]^n}{(p-1)s^2} g(\tilde{\rho}, \mu, \nu) \\ &= \frac{[p(p+1)/2]^n}{(p-1)s^2} \sum_{0 \leq \mu, \nu < s} \sum_{1 \leq \tilde{\rho} < p} g(\tilde{\rho}, \mu, \nu) \\ &= \frac{[p(p+1)/2]^n}{(p-1)s^2} \cdot s^2 \cdot \frac{r(r+1)}{2}. \quad \square \end{aligned}$$

## 6. HAUSDORFF DIMENSION OF $C(m, n) \pmod{p}$

A “fractal set” corresponding to the pattern of all nonzero residues of the generalized binomial coefficients modulo a prime  $p$  is constructed as a subset of the square  $[0, 1) \times [0, 1)$  by “tremas” as follows. We combine all the nonzero residues because the construction below for a fixed residue will not always yield nested sets. Flath and Peele [5] give an alternative, rescaled lattice construction.

For each  $k$  let  $\mathcal{G}_k$  denote the class of sets

$$G_{m, n}^{(k)} = \bigcup_{\substack{0 \leq i, j < r \\ i+j < r}} \left[ \frac{mr+i}{rp^k}, \frac{mr+i+1}{rp^k} \right) \times \left[ \frac{nr+j}{rp^k}, \frac{nr+j+1}{rp^k} \right)$$

with  $0 \leq m, n < p^k$  and  $p \nmid B(m, n)$ , and let  $G_k$  be their union. Proposition 1 and Lucas's theorem imply that  $G_{m,n}^{(k)}$  is contained in some set in  $\mathcal{G}_{k-1}$  and contains a finite number of disjoint sets of  $\mathcal{G}_{k+1}$ , and  $G_{k+1} \subset G_k$ . Accordingly our fractal set is

$$G := \bigcap_{k \in \mathbb{N}} G_k.$$

Figure 1 shows a density plot of a pre-fractal image of the fibonomial residues modulo 3; the nonwhite squares are components of  $G_3$ .

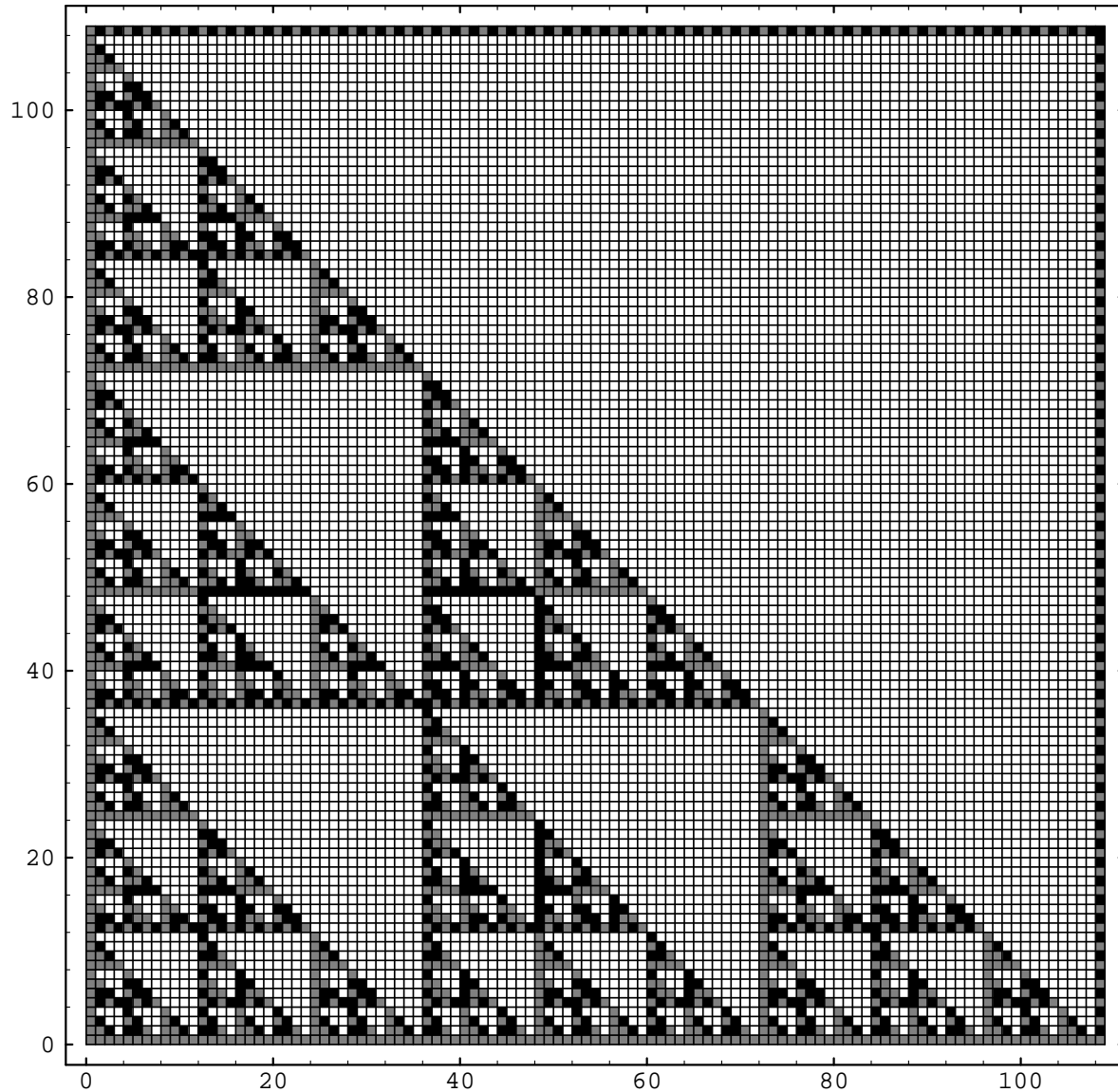


FIGURE 1. Fibonomial coefficients mod 3

**Theorem 2.** *If  $p$  is a prime that does not divide  $b$ , then the fractal set  $G$  constructed above has Hausdorff dimension*

$$\dim_H(G) = \frac{\log \binom{p+1}{2}}{\log p}.$$

**Proof.** The proof uses (1) the fact [3, p. 43] that

$$\dim_H G \leq \dim_B G,$$

where  $\dim_B G$  is the box-counting dimension of  $G$ , and (2) the mass distribution principle [3, p. 55]: if  $\mu$  is a measure on a set  $F$  and for some  $s$  there are numbers  $c > 0, \delta > 0$  such that

$$\mu(U) \leq c|U|^s$$

for all sets  $U$  with  $|U| \leq \delta$  (where  $|U|$  is the diameter of  $U$ ), then the Hausdorff measure  $\mathcal{H}^s(F) \geq \mu(F)/c$  and  $s \leq \dim_H(F)$ . The box-counting dimension of  $G$  may be calculated [3, p. 41] by the formula

$$\dim_B G = \lim_{k \rightarrow \infty} \frac{\log N_{\delta_k}(G)}{-\log \delta_k},$$

where  $N_{\delta}(G)$  is the smallest number of  $\delta$ -mesh squares that intersect the set  $G$ , provided that the sequence  $(\delta_k)$  decreases to zero and  $\delta_{k+1} \geq \eta \delta_k$  for some positive constant  $\eta$ . Let us choose  $\delta_k = 1/(rp^k)$  (and  $\eta = 1/p$ ). Then

$$N_{\delta_k}(G) = \frac{r(r+1)}{2} \left[ \frac{p(p+1)}{2} \right]^k,$$

so

$$\begin{aligned} \dim_B(G) &= \lim_{k \rightarrow \infty} \frac{\log N_{\delta_k}(G)}{-\log \delta_k} \\ &= \lim_{k \rightarrow \infty} \frac{\log \left( \frac{r(r+1)}{2} \left[ \frac{p(p+1)}{2} \right]^k \right)}{-\log[1/(rp^k)]} \\ &= \lim_{k \rightarrow \infty} \frac{\log \frac{r(r+1)}{2} + k \log \frac{p(p+1)}{2}}{\log r + k \log p} \\ &= \frac{\log \binom{p+1}{2}}{\log p}. \end{aligned}$$

Now let  $\mu$  be the ‘‘natural measure’’ defined by repeated subdivision [3, pp. 13–14] that assigns weight  $\binom{p+1}{2}^{-k}$  to each set in  $\mathcal{G}_k$  and weight 0 to the complement of  $G_k$ : At stage  $k+1$ , the weight of each  $G_{m,n}^{(k)}$  is evenly divided among the  $\binom{p+1}{2}$  sets in  $\mathcal{G}_{k+1}$  contained therein. We shall see that there exist  $c > 0$  and  $\delta > 0$  such that

$$\mu(U) \leq c|U|^d \quad \text{where} \quad d := \frac{\log \binom{p+1}{2}}{\log p}$$

for all sets  $U$  with diameter  $|U| \leq \delta$ . Let  $\delta \in (0, 1)$ . Suppose  $|U| \leq \delta$ . Let  $k$  be the integer such that  $1/p^{k+1} \leq |U| < 1/p^k$ . Note that then  $1/p^k \leq p|U|$  and  $U$  meets at

most four of the sets in  $\mathcal{G}_k$  (because  $U$  is contained in a square of side  $|U|$  with sides parallel the coordinate axes, and this containing square can intersect no more than four  $G_{m,n}^{(k)}$ 's). Therefore,

$$\begin{aligned} \mu(U) &\leq 4 \frac{1}{\binom{p+1}{2}^k} \\ &= \frac{4}{(p^d)^k} \left[ \text{because } d = \frac{\log \binom{p+1}{2}}{\log p} \right] \\ &= 4 \left( \frac{1}{p^k} \right)^d \\ &\leq 4(p|U|)^d, \end{aligned}$$

so  $\mu(U) \leq c|U|^d$  for all sets  $U$  with  $|U| \leq \delta$  where  $c = 4p^d$ . By the mass distribution principle,  $d \leq \dim_H G$ . But from before,  $\dim_B G = d$ , and we know  $\dim_H G \leq \dim_B G$ , so we must have  $\dim_H G = d = \log \binom{p+1}{2} / \log p$ .  $\square$

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