FRACTAL DIMENSION OF GENERALIZED MULTINOMIAL COEFFICIENTS MODULO A PRIME: PRELIMINARY REPORT

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Abstract. Given a sequence (u_n) of positive integers generated by $u_1 = 1, u_2 = a, u_n = au_{n-1} + bu_{n-2} (n \geq 3)$, define the generalized factorial by $[n]! = u_1 u_2 \cdots u_n$ and the generalized d-nomial coefficient by $C(n_1, \ldots, n_d) = [n_1 + \cdots + n_d]!/([n_1]! \cdots [n_d]!)$. Assume that the prime p does not divide b. Let $r = \min\{n : p|u_n\}$.

Theorem 1 (Asymptotic abundance of residues): $\#\{(n_1, \ldots, n_d)|0 \leq n_1, \ldots, n_d < rp^k \text{ and } C(n_1, \ldots, n_d) \equiv \rho (\text{mod } p)\} \sim \frac{1}{p-1}\binom{r+d-1}{d}\binom{p+d-1}{d}^k$ as $k \to \infty$ for $\rho = 1, \ldots, p-1$.

Theorem 2 (Fractal dimension): Let $s_k = rp^k$. The Hausdorff dimension of $\cap_k \cup \{[n_1/s_k, (n_1 + 1)/s_k) \times \cdots \times [n_d/s_k, (n_d + 1)/s_k)|0 \leq n_1, \ldots, n_d < s_k, p \nmid C(n_1, \ldots, n_d)\}$ is $\log\binom{p+d-1}{d}/\log p$.

Lucas's Theorem and Pascal's Triangle Modulo p

A remarkable theorem of E. Lucas [19] expresses the binomial coefficient $\binom{N}{m}$ modulo a prime p in terms of the binomial coefficients of the base-p digits of N and m: If $N = \sum N_j p^j$ and $m = \sum m_j p^j$ where $0 \leq N_j, m_j < p$, then

$$\binom{N}{m} \equiv \prod \binom{N_j}{m_j} \pmod{p}.$$

Alternatively, if we let

$$B(m, n) = \binom{m+n}{m} = \frac{(m+n)!}{m!n!},$$

then we have the recursive formula

$$B(m, n) \equiv B(m \div p, n \div p) B(m \bmod p, n \bmod p) \pmod{p}$$

where $m \div p$ is the integer quotient of m by p, and $m \bmod p$ is the remainder. Then if $m = \sum m_j p^j$ and $n = \sum n_j p^j$ where $0 \leq m_j, n_j <$

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p, then

$$B(m,n) \equiv \prod B(m_j,n_j) \pmod{p}.$$

This theorem implies that the residues of Pascal's triangle modulo p have a self-similar structure; see, e.g., [21], [2], [9], [10], [18], [26], and [1]. For example, if $p = 3$, then the matrix $[B(m,n) \bmod p]$ for $0 \le m,n < 9$ is given as follows:

$$\begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ 1&2&0&1&2&0&1&2&0 \\ 1&0&0&1&0&0&1&0&0 \\ 1&1&1&2&2&2&0&0&0 \\ 1&2&0&2&1&0&0&0&0 \\ 1&0&0&2&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0 \\ 1&2&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \end{bmatrix} \equiv \begin{bmatrix} 1\mathbf{B} & 1\mathbf{B} & 1\mathbf{B} \\ 1\mathbf{B} & 2\mathbf{B} & 0\mathbf{B} \\ 1\mathbf{B} & 0\mathbf{B} & 0\mathbf{B} \end{bmatrix} \pmod{p},$$

where

$$\mathbf{B} = \begin{bmatrix} 1&1&1 \\ 1&2&3 \\ 1&3&6 \end{bmatrix} \equiv \begin{bmatrix} 1&1&1 \\ 1&2&0 \\ 1&0&0 \end{bmatrix} \pmod{p},$$

so this matrix is the tensor (or, Kronecker) product $\mathbf{B} \otimes \mathbf{B} \bmod p$, Generally, as noted in [20], modulo p we have that $[B(m,n) \bmod p]$ for $0 \le m,n < p^k$ will be $\mathbf{B}^{\otimes k}$, the k-fold tensor product of $\mathbf{B} = [B(i,j) \bmod p]$ where $0 \le i,j < p$. Note that matrix indices start at index pair $(0,0)$.

If such an array of binomial coefficients is color coded (or gray-scale coded), the resulting patterns are quite stunning. See the accompanying figure for an example.

Wolfram [26] and Flath and Peele [7] have determined that the "fractal" dimension of geometric representations of the pattern of the nonzero residues of Pascal's triangle is given by $\log \binom{p+1}{2} / \log p$. The purpose of this paper is to prove similar fractal dimension results, as well as density results, for a large class of generalized multinomial coefficients.

Multinomial Coefficients Modulo p

L.E. Dickson [4, p. 273], [5] generalized Lucas's theorem to multinomial coefficients. From here on, let d denote a fixed integer that is greater than or equal to 2. Let

$$M(n_1,n_2,\dots,n_d) := \binom{n_1+n_2+\cdots n_d}{n_1,n_2,\dots,n_d} = \frac{(n_1+n_2+\cdots n_d)!}{n_1!n_2!\dots,n_d!}.$$

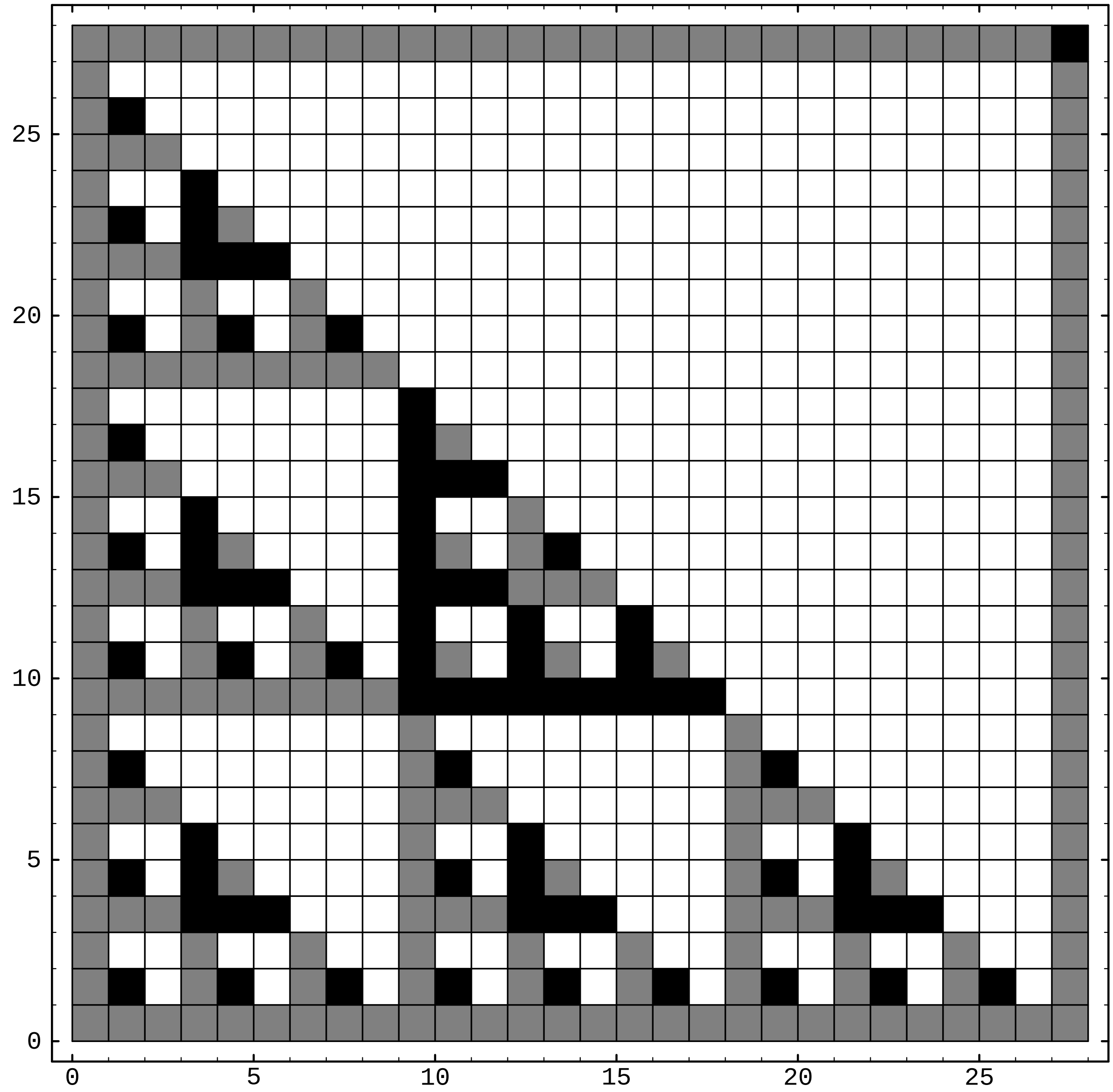

FIGURE 1. Binomial coefficients mod 3

If each $n_i = \sum_j n_{ij} p^j$ where $0 \le n_{ij} < p$, then

$$M(n_1, n_2, \ldots, n_d) \equiv \prod_j M(n_{1j}, n_{2j}, \ldots, n_{dj}) \pmod p.$$

The recursive formula is:

$$M(n_1, n_2, \ldots, n_d) \equiv M(n'_1, \ldots, n'_d) M(n_{1,0}, \ldots, n_{d,0}) \pmod p,$$

where each $n'_i = n_i \div p$ and $n_{i,0} = n_i \bmod p$.

A graphical illustration of Dickson's theorem is shown in the accompanying figure.

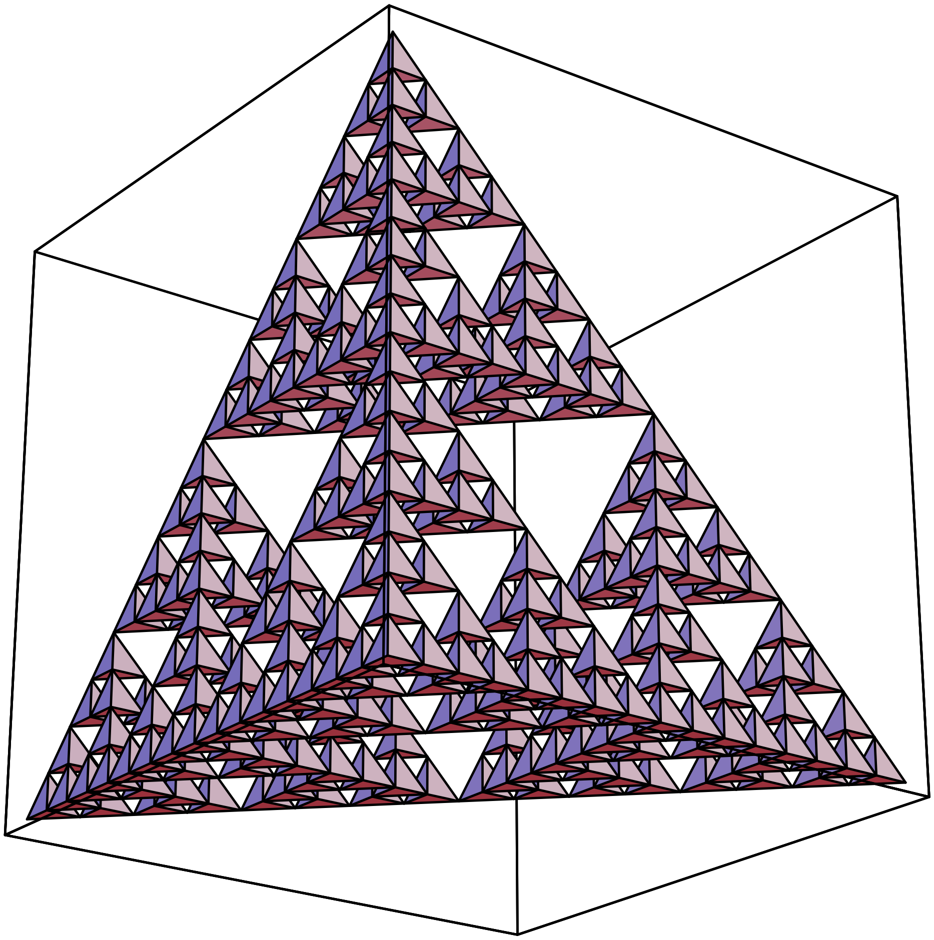

Figure 2. Multinomial coefficients mod 2

Generalized Multinomial Coefficients

Generalized d-nomial coefficients corresponding to a given sequence (u_k) are defined analogously to $M(n_1, \ldots, n_d)$ by replacing $n!$ by the product of u_1 through u_n,

$$[n]! := \prod_{j=1}^{n} u_j,$$

and then defining

$$C(n_1, n_2, \dots, n_d) = \frac{(n_1 + n_2 + \cdots + n_d)!}{n_1!n_2!\cdots n_d!}$$

(assuming any zero factors in the numerator and denominator are first paired and then cancelled).

In this paper we assume that the sequence is defined by a second-order recurrence relation as follows:

$$u_0 = 0; u_1 = 1; u_n = au_{n-1} + bu_{n-2} \text{ for } n = 2, 3, 4, \dots$$

where a and b are integers.

When $a = 2$ and $b = -1$, then $u_k = k$ and the generalized d-nomial coefficients become the ordinary multinomial coefficients: $C(n_1, n_2, \dots, n_d) = (n_1 + n_2 + \cdots + n_d)!/[n_1!n_2!\cdots n_d!)]$. When $a = 1 + q$ and $b = -q$, then $u_k = 1 + q + q^2 + \cdots + q^{k-1}$ and the generalized 2-nomial coefficients are the Gauss q-binomial coefficients. When $a = 1$ and $b = 1$, then $u_k = F_k$, the k^{th} Fibonacci number, and the generalized 2-nomial coefficients become the fibonomial coefficients.

Residues of Generalized Multinomial Coefficients

Wells [22] [23] has proved a generalization of the Lucas theorem for generalized 2-nomial coefficients. An alternative version is given in [13], and its proof may be generalized to provide the extension of Dickson's theorem to generalized d-nomial coefficients that we need for the purposes of our density and fractal dimension calculations.

Definition 1. Let r denote the rank of apparition of p; thus, $r = \min\{n \in \mathbb{N} : u_n \equiv 0 \pmod{p}\}$. Let t denote the (least) period of $\langle u_n \bmod p\rangle$, if it exists. Let $s = t/r$.

Notation: If $r < \infty$, then for each nonnegative integer n, let

$$\begin{aligned} n_0 &= n \bmod r, \\ n' &= n \div r, \\ n^* &= n \bmod t, \\ n'' &= n^* \div r = n' \bmod s. \end{aligned}$$

The following lemmata may be proved by generalizing the ideas in [13]

Lemma 1. *Addition formula:*

$$u_{n_1+\cdots+n_d} = \sum_{i=1}^{d} u_{n_i} \left(\prod_{j<i} b u_{n_j-1} \right) u_{1+\sum_{j>i} n_j}.$$

Lemma 2. *Reduction formula: For* $n_1 \geq 1, \ldots, n_d \geq 1$,

$$C(n_1, \ldots, n_d) = \sum_{i=1}^{d} \left(\prod_{j<i} b u_{n_j-1} \right) u_{1+\sum_{j>i} n_j} C(n_1, \ldots, n_i - 1, \ldots, n_d).$$

Lemma 3. *r-step recurrence mod p:*

$$C(n_1 r, \ldots, n_d r) \equiv \sum_{i=1}^{d} u_{r+1}^{r \sum_{j \neq i} n_j} C(n_1 r, \ldots, (n_i - 1) r, \ldots, n_d r) \quad \textit{(mod } p\textit{)}.$$

Lemma 4.

$$C(n_1 r, \ldots, n_d r) \equiv u_{r+1}^{r \sum_{i<j} n_i n_j} M(n_1, \ldots, n_d) \quad \textit{(mod } p\textit{)}.$$

Definition 2. For $n_1, \ldots, n_d \geq 0$ and $0 \leq n_{1,0}, \ldots, n_{d,0} < r$, let $A(n_1, \ldots, n_d; n_{1,0}, \ldots, n_{d,0})$ denote the (unique) solution of the modulo-p recurrence relation

$$A(n_1, \ldots, n_d; n_{1,0}, \ldots, n_{d,0}) \equiv$$
$$\sum_{i=1}^{d} \left(\prod_{j<i} b u_{n_j r + n_{j,0} - 1} \right) u_{1+\sum_{j>i} (n_j r + n_{j,0})} A(n_1, \ldots, n_d; n_{1,0}, \ldots, n_{i,0} - 1, \ldots, n_{d,0})$$

satisfying the boundary conditions

$$A(n_1, \ldots, n_d; n_{1,0}, \ldots, n_{d,0}) \equiv 0 \pmod{p}$$

when $n_{i,0} = -1$ and $1 \leq n_{j,0} < r$ for $j \neq i$, for $i = 1, \ldots, d$, and

$$A(n_1, \ldots, n_d; 0, \ldots, 0) \equiv 1 \pmod{p}.$$

Note that if the final recurrence in this definition were

$$A(n_1, \ldots, n_d; 0, \ldots, 0) \equiv C(n_1 r, \ldots, n_d r) \pmod{p},$$

then the solution would be

$$A(n_1, \ldots, n_d; n_{1,0}, \ldots, n_{d,0}) \equiv C(n_1 r + n_{1,0}, \ldots, n_d r + n_{d,0}) \pmod{p}$$

for $0 \leq n_{1,0}, \ldots, n_{d,0} < r$. As a consequence of this note and linearity,

$$C(n_1, \ldots, n_d) \equiv C(n'_1 r, \ldots, n'_d r) A(n'_1, \ldots, n'_d; n_{1,0}, \ldots, n_{d,0}) \pmod{p} \tag{1}$$

where $n'_i = n_i \div r$ and $n_{i,0} = n_i \bmod r$.

Because the sequence $(u_k \bmod p)$ has period t, we have $n'_i r \equiv n''_i r \pmod{t}$ for $i = 1, \ldots, d$, where $n''_i := (n_i \bmod t) \div r$, whence

$$u_{rn'_j+n_{j,0}} \equiv u_{rn''_j+n_{j,0}} \pmod{p} \text{ for } j = 1, \ldots, d$$

and

$$u_{1+\sum_{j>i}(rn'_j+n_{j,0})} \equiv u_{1+\sum_{j>i}(rn''_j+n_{j,0})} \pmod{p} \text{ for } i = 1, \ldots, d,$$

so the coefficients in the recurrence formula for A are congruent to those with each n'_j replaced by u''_j. Therefore,

$$A(n'_1, \ldots, n'_d; n_{1,0}, \ldots, n_{d,0}) \equiv A(n''_1, \ldots, n''_d; n_{1,0}, \ldots, n_{d,0}) \pmod{p}, \tag{2}$$

and, since $u^n_{r+1} \equiv u_{nr+1} \pmod{p}$, we alos have

$$u_{r+1}^{r\sum_{i<j} n'_i n'_j} \equiv u_{r+1}^{r\sum_{i<j} n''_i n''_j} \pmod{p}, \tag{3}$$

Combining equations (1) and (2), we get

$$C(n_1, \ldots, n_d) \equiv C(n'_1 r, \ldots, n'_d r) A(n''_1, \ldots, n''_d; n_{1,0}, \ldots, n_{d,0}) \pmod{p}.$$

Then, by Lemma 4 and equation (3), we obtain a generalization of Dickson's theorem.

$$C(n_1, \ldots, n_d) \equiv M(n'_1, \ldots, n'_d) u_{r+1}^{r\sum_{i<j} n''_i n''_j} A(n''_1, \ldots, n''_d; n_{1,0}, \ldots, n_{d,0}) \pmod{p}. \tag{4}$$

Definition 3. If $r < \infty$, then for $m_1, \ldots, m_d \geq 0$ and $0 \leq n_1, \ldots, n_d < r$, define

$$H_{m_1,\ldots,m_d}(n_1, \ldots, n_d) = u_{r+1}^{r\sum_{i<j} m_i m_j} A(m_1, \ldots, m_d; n_{1,0}, \ldots, n_{d,0}).$$

Then $H_{n'_1,\ldots,n'_d}(n_{1,0}, \ldots, n_{d,0}) \equiv H_{n''_1,\ldots,n''_d}(n_{1,0}, \ldots, n_{d,0}) \pmod{p}$.

Thus, equation (4) may be written in the form we shall use:

Theorem 1.

$$C(n_1, \ldots, n_d) \equiv M(n'_1, \ldots, n'_d) H_{n''_1,\ldots,n''_d}(n_{1,0}, \ldots, n_{d,0}) \quad (mod\ p).$$

This result simplifies nicely when $s = 1$. Then each $n''_i = 0$, and $H_{0,\ldots,0}(n_{1,0}, \ldots, n_{d,0}) \equiv C(n_{1,0}, \ldots, n_{d,0}) \pmod{p}$ for $0 \leq n_{1,0}, \ldots, n_{d,0} < r$.

Corollary 1. *If $p \nmid b$ and $s = 1$, then, for $n_1, \ldots, n_d \geq 0$,*

$$C(n_1, \ldots, n_d) \equiv M(n'_1, \ldots, n'_d) C(n_{1,0}, \ldots, n_{d,0}) \ (mod\ p).$$

Thus, in this case, as in the Pascal "triangle" case, the pattern of residues exhibits self-similarity upon scaling by p by Dickson's theorem for multinomial coefficients.

EXAMPLES

The following examples are borrowed from [11].

Example 1: q-binomial coefficients. Take $u_n = \sum_{k=0}^{n-1} q^k$ to obtain the q-binomial coefficients. For a numerical example, take $q = 2$ and $p = 5$. Then $u_1 = 1, u_2 = 3, u_3 = 7, u_4 = 15, u_5 = 31, \ldots,$ whence $r = 4$, and

$$\mathbf{C}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 7 & 15 \\ 1 & 7 & 35 & 155 \\ 1 & 15 & 155 & 1395 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \pmod 5,$$

so

$$\mathbf{C}_1 \equiv \mathbf{B} \otimes \mathbf{C}_0 \equiv \begin{bmatrix} 1\mathbf{C}_0 & 1\mathbf{C}_0 & 1\mathbf{C}_0 & 1\mathbf{C}_0 & 1\mathbf{C}_0 \\ 1\mathbf{C}_0 & 2\mathbf{C}_0 & 3\mathbf{C}_0 & 4\mathbf{C}_0 & 0\mathbf{C}_0 \\ 1\mathbf{C}_0 & 3\mathbf{C}_0 & 1\mathbf{C}_0 & 0\mathbf{C}_0 & 0\mathbf{C}_0 \\ 1\mathbf{C}_0 & 4\mathbf{C}_0 & 0\mathbf{C}_0 & 0\mathbf{C}_0 & 0\mathbf{C}_0 \\ 1\mathbf{C}_0 & 0\mathbf{C}_0 & 0\mathbf{C}_0 & 0\mathbf{C}_0 & 0\mathbf{C}_0 \end{bmatrix} \pmod 5.$$

Example 2: Fibonomial coefficients modulo p. Let $a = b = 1$ so that $u_n = F_n$, and consider the case $p = 3$. Then $r = 4$, $t = 8$, and $s = 2$. The initial part of the table of fibonomial coefficients modulo 3 is given in Table 1.

By Definition 3,

$$\mathbf{H}_{0,0} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \mathbf{H}_{0,1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}; \mathbf{H}_{1,0} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \mathbf{H}_{1,1} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}.$$

The structure of the matrix of fibonomial coefficients modulo 3, in accordance with Theorem 1, is given in Table 2. Wells [25] provides a detailed description of the pattern of these submatrices from a "triangular" perspective. The pattern of fibonomial coefficients modulo any prime is given in [11].

Theorem 1 and the example show that the infinite matrix $[C(i,j) \bmod p]$ may be partitioned into $r \times r$ submatrices which form basic, natural "tiling units." The pattern of the residues is obtained by superimposing the self-similar array of binomial coefficients modulo p upon the doubly periodic "tiling" of the plane by "hidden" $r \times r$ $\mathbf{H}$ matrices. The binomial structure is self-similar upon scaling by the factor p. The $r \times r$ tiling structure has period s both horizontally and vertically, and

1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1
1 1 2 0	2 2 1 0	1 1 2 0	2 2 1 0	1 1 2 0	2 2 1 0	1 1 2 0	2 2 1 0	1 1 2 0
1 2 0 0	1 2 0 0	1 2 0 0	1 2 0 0	1 2 0 0	1 2 0 0	1 2 0 0	1 2 0 0	1 2 0 0
1 0 0 0	2 0 0 0	1 0 0 0	2 0 0 0	1 0 0 0	2 0 0 0	1 0 0 0	2 0 0 0	1 0 0 0
1 2 1 2	2 1 2 1	0 0 0 0	1 2 1 2	2 1 2 1	0 0 0 0	1 2 1 2	2 1 2 1	0 0 0 0
1 2 2 0	1 2 2 0	0 0 0 0	2 1 1 0	2 1 1 0	0 0 0 0	1 2 2 0	1 2 2 0	0 0 0 0
1 1 0 0	2 2 0 0	0 0 0 0	1 1 0 0	2 2 0 0	0 0 0 0	1 1 0 0	2 2 0 0	0 0 0 0
1 0 0 0	1 0 0 0	0 0 0 0	2 0 0 0	2 0 0 0	0 0 0 0	1 0 0 0	1 0 0 0	0 0 0 0
1 1 1 1	0 0 0 0	0 0 0 0	1 1 1 1	0 0 0 0	0 0 0 0	1 1 1 1	0 0 0 0	0 0 0 0
1 1 2 0	0 0 0 0	0 0 0 0	2 2 1 0	0 0 0 0	0 0 0 0	1 1 2 0	0 0 0 0	0 0 0 0
1 2 0 0	0 0 0 0	0 0 0 0	1 2 0 0	0 0 0 0	0 0 0 0	1 2 0 0	0 0 0 0	0 0 0 0
1 0 0 0	0 0 0 0	0 0 0 0	2 0 0 0	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0
1 2 1 2	1 2 1 2	1 2 1 2	2 1 2 1	2 1 2 1	2 1 2 1	0 0 0 0	0 0 0 0	0 0 0 0
1 2 2 0	2 1 1 0	1 2 2 0	1 2 2 0	2 1 1 0	1 2 2 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 0 0	1 1 0 0	1 1 0 0	2 2 0 0	2 2 0 0	2 2 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 0 0 0	2 0 0 0	1 0 0 0	1 0 0 0	2 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 1 1	2 2 2 2	0 0 0 0	2 2 2 2	1 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 2 0	1 1 2 0	0 0 0 0	1 1 2 0	1 1 2 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 0 0	2 1 0 0	0 0 0 0	2 1 0 0	1 2 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 0 0 0	1 0 0 0	0 0 0 0	1 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 1 2	0 0 0 0	0 0 0 0	2 1 2 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 2 0	0 0 0 0	0 0 0 0	1 2 2 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 0 0	0 0 0 0	0 0 0 0	2 2 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 0 0 0	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 1 1	1 1 1 1	1 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 2 0	2 2 1 0	1 1 2 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 0 0	1 2 0 0	1 2 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 0 0 0	2 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 1 2	2 1 2 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 2 0	1 2 2 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 0 0	2 2 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 2 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0

TABLE 1. The Fibonomial coefficients modulo 3

$$\begin{bmatrix}
1\mathbf{H}_{0,0} & 1\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & 1\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & 1\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & 1\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & \cdots \\
1\mathbf{H}_{1,0} & 2\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 1\mathbf{H}_{1,1} & 2\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 1\mathbf{H}_{1,0} & 2\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & \cdots \\
1\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 1\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & \cdots \\
1\mathbf{H}_{1,0} & 1\mathbf{H}_{1,1} & 1\mathbf{H}_{1,0} & 2\mathbf{H}_{1,1} & 2\mathbf{H}_{1,0} & 2\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & \cdots \\
1\mathbf{H}_{0,0} & 2\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 2\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & \cdots \\
1\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 2\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & \cdots \\
1\mathbf{H}_{0,0} & 1\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & \cdots \\
1\mathbf{H}_{1,0} & 2\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & \cdots \\
1\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots
\end{bmatrix}$$

TABLE 2. Submatrices of the fibonomial coefficients mod 3

so the period is t at the element level. When $s = 1$, there are $p-1$ different nonzero $r \times r$ submatrices, one for each nonzero residue value of $B(m', n')$ mod p times $\mathbf{C}_0$. In the general case, there are also $s \cdot s$ different $H_{m'',n''}$-matrices. In fact, there are $(p-1)s^2$ different nonzero "tiles" [13].

In the case of the the fibonomial coefficients modulo 3, the exhibited matrix shows these seven submatrices:

$$1\mathbf{H}_{0,0}, 1\mathbf{H}_{0,1}, 1\mathbf{H}_{1,0}, 1\mathbf{H}_{1,1}, 2\mathbf{H}_{0,1}, 2\mathbf{H}_{1,0}, 2\mathbf{H}_{1,1}.$$

The missing case, $2\mathbf{H}_{0,0}$, must be sought farther out. The places of the missing $2\mathbf{H}_{0,0}$ are $(5, 11), (11, 5), (5, 13), (13, 5) \ldots$ in Table 2.

A density plot of the initial 37 array of Fibonomial coefficients modulo 3 is shown in the accompanying figure.

Example 3: 3-Fibonomial Coefficients When $d = 3$ and the sequence (u_k) is the Fibonacci sequence, then the generalized d-nomial coefficients are the 3-Fibonomials. The self-similarity of the 3-dimensional array of 3-Fibonomials not congruent to 0 modulo p is suggested in the accompanying figures.

SCALING-UP RECURSION FORMULA

Define

$$C_{\alpha_1,\ldots,\alpha_d}(n_1, \ldots, n_d) \equiv M(n'_1, \ldots, n'_d) H_{\alpha_1+n''_1,\ldots,\alpha_d+n''_d}(n_1, \ldots, n_d) \pmod{p}.$$

Using Theorem 1 and the fact that $s|p-1$, we may deduce the following recursion relation.

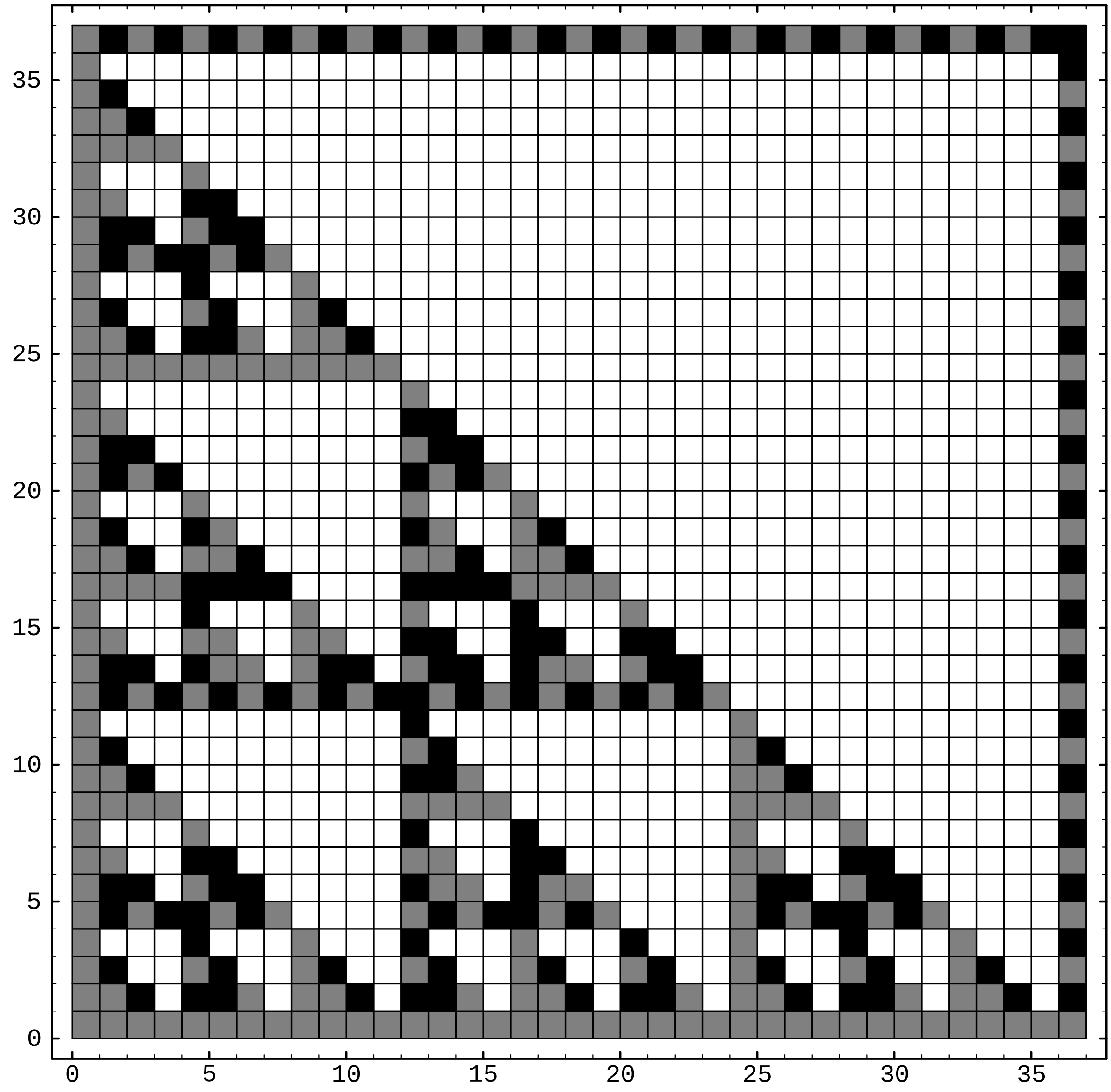

FIGURE 3. 2-Fibonomials mod 3

Proposition 1. *If for each* i, $n_i = n_{i,k}p^{k-1}r + n_i^{(k)}$ *where* $0 \le n_i^{(k)} < rp^{k-1}$, *then*

$$C_{\alpha_1,\dots,\alpha_d}(n_1,\dots,n_d) \equiv M(n_{1,k},\dots,n_{d,k})C_{\alpha_1+n_{1,k},\dots,\alpha_d+n_{d,k}}(n_1^{(k)},\dots,n_d^{(k)}) \pmod p.$$

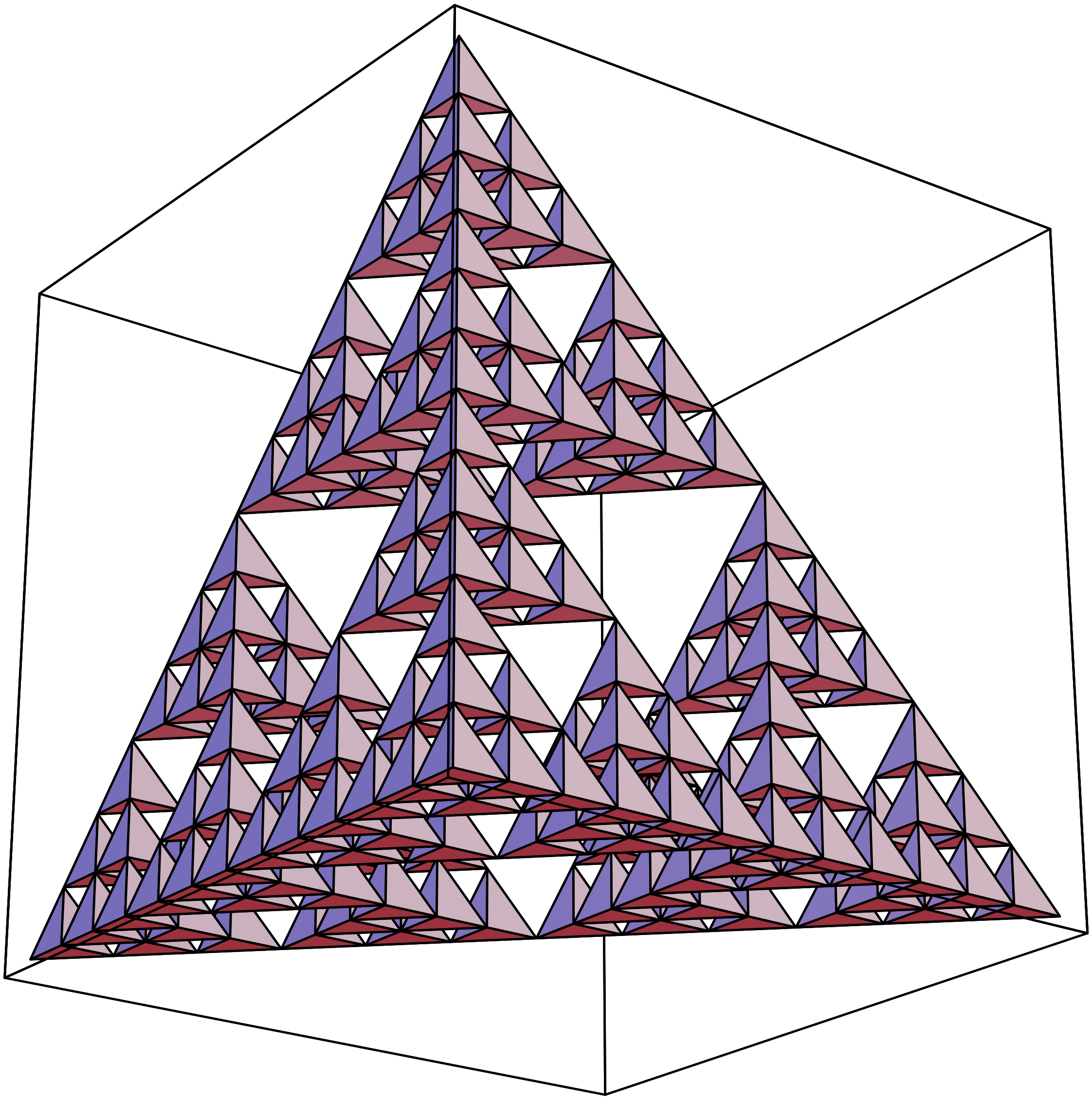

Figure 4. 3-Fibonomials mod 2

Asymptotic Abundance of Residues

Also define

$$\mathbb{C}^{(k)}_{\alpha_1,\dots,\alpha_d} = [C_{\alpha_1,\dots,\alpha_d}(n_1,\dots,n_d)] \qquad (0 \le n_1,\dots,n_d < rp^k),$$

$$\mathbb{C}^{(k)} = \mathbb{C}_{0,\dots,0},$$

$$\begin{aligned} f^{(k)}_{\alpha_1,\dots,\alpha_d}(\rho,\nu_1,\dots,\nu_d) = \#\{&(m_1,\dots,m_d) : 0 \le m_1,\dots,m_d < p^k, \\ &C_{\alpha_1,\dots,\alpha_d}(m_1 r + m_{1,0},\dots,m_d r + m_{d,0}) \equiv \rho H_{\mu_1,\dots,\mu_d}(m_{1,0},\dots,m_{d,0}) \\ &\text{for } 0 \le m_{1,0},\dots,m_{d,0} < r\}, \end{aligned}$$

$$f^{(k)}(\rho,\nu,\dots,\nu_d) = f^{(k)}_{0,\dots,0}(\rho,\nu_1,\dots,\nu_d).$$

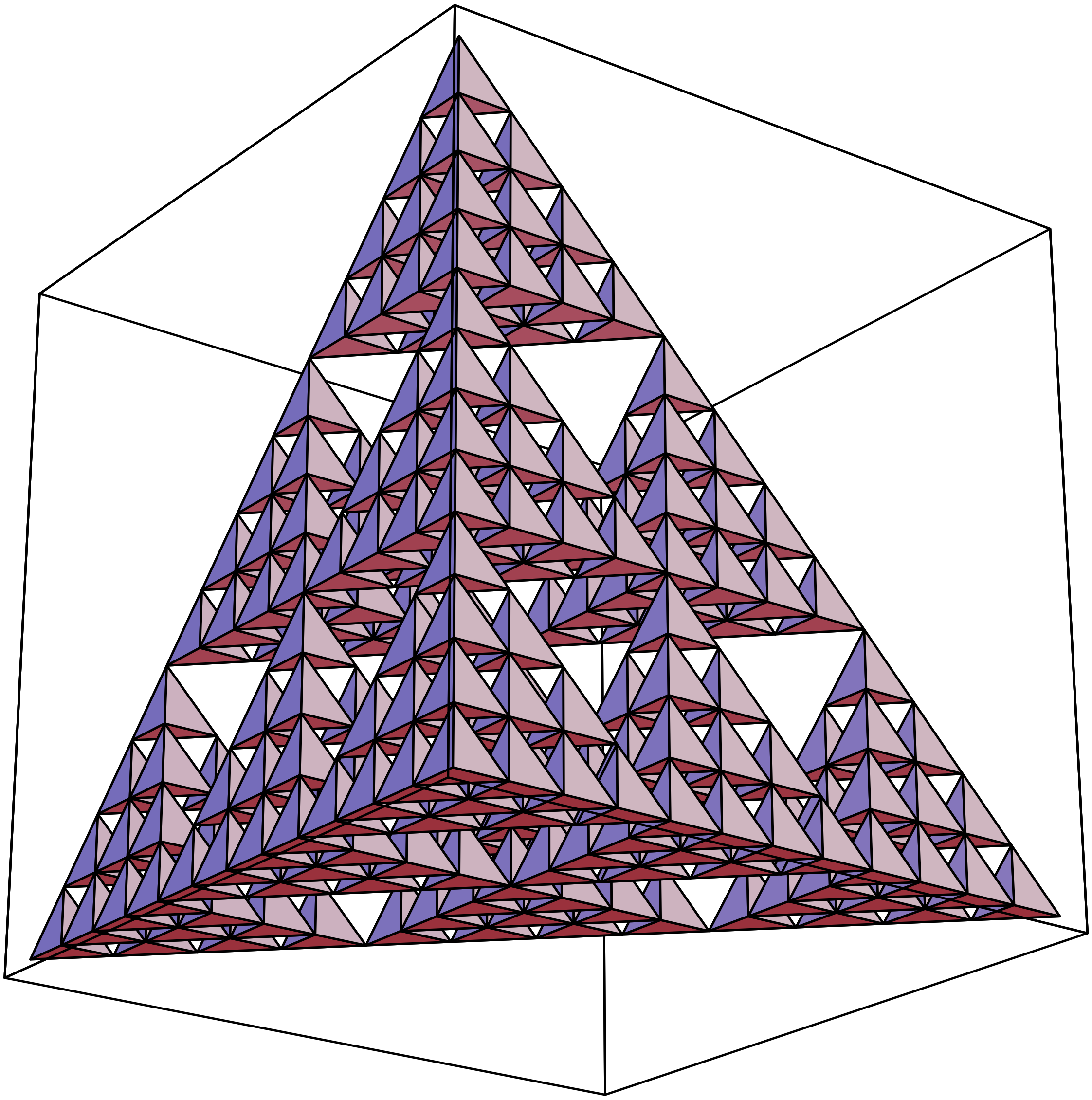

FIGURE 5. 3-Fibonomials mod 3: nonzero residues

The quantity $f^{(k)}(\rho, \nu, \ldots, \nu_d)$ is our focus for now. It is the number of $\rho\mathbb{H}_{\nu_1,\ldots,\nu_d}$ blocks in the initial $rp^k \times \cdots \times rp^k$ block of $C(n_1, \ldots, n_d)$ values.

Lemma 5.

$$f^{(k)}_{\alpha_1,\ldots,\alpha_d}(\rho, \nu_1, \ldots, \nu_d) = f^{(k)}(\rho, \nu_1 - \alpha_1, \ldots, \nu_d - \alpha_d).$$

Lemma 6. *For* $1 \le \rho < p$ *and* $0 \le \nu_1, \dots, \nu_d < s$,

$$f^{(k)}(\rho, \nu_1, \dots, \nu_d) = \sum_{\substack{0 \le \mu_1, \dots, \mu_d < p \\ \mu_1 + \dots + \mu_d < p}} f^{(k-1)}(M(\mu_1, \dots, \mu_d)^{-1}\rho, \nu_1 - \mu_1, \dots, \nu_d - \mu_d).$$

Letting $I = (\rho, \nu_1, \dots, \nu_d)$ and $J = (\tilde{\rho}, \tilde{\nu}_1, \dots, \tilde{\nu}_d)$, define

$$\begin{aligned} Q^I_J = \#\{(\mu_1, \dots, \mu_d) :& 0 \le \mu_1, \dots, \mu_d < p, \quad \mu_1 + \dots + \mu_d < p, \\ & M(\mu_1, \dots, \mu_d)^{-1}\rho \equiv \tilde{\rho} \pmod{p}, \\ & \nu_1 - \mu_1 \equiv \tilde{\nu}_1 \pmod{s}, \dots \\ & \nu_d - \mu_d \equiv \tilde{\nu}_d \pmod{s}\} \end{aligned}$$

and

$$\mathbb{Q} = [Q^I_J].$$

Lemma 7.

$$\sum_J Q^I_J = \binom{p+d-1}{d} \quad \textit{and} \quad \sum_I Q^I_J = \binom{p+d-1}{d}.$$

Accordingly, let

$$P^I_J = \binom{p+d-1}{d}^{-1} Q^I_J.$$

Then

$$\mathbb{P} := [P^I_J]$$

is a doubly stochastic matrix, and so its stationary vector is the constant vector consisting of $(p-1)s^d$ entries all equal to $1/((p-1)s^d)$.

Lemma 8. *Every entry of* $\mathbb{Q}^{d+1}$ *is positive, and so is every entry of* $\mathbb{P}^{d+1}$. *If* $s = 1$, *all entries of* $\mathbb{Q}$ *and* $\mathbb{P}$ *are already positive. Consequently, the matrix* $\mathbb{Q}$ *is primitive, and the finite Markov chain having* $\mathbb{P}$ *as its transition matrix is regular.*

Lemma 9. *Regarding* $f^{(k)}$ *as a row vector with indices* $I = (\rho, \nu_1, \dots, \nu_d)$, *we have*

$$f^{(k)} = f^{(0)}\mathbb{Q}^k.$$

Theorem 2. *For* $1 \le \rho < p$ *and* $0 \le \nu_1, \dots, \nu_d < s$,

$$f^{(k)}(\rho, \nu_1, \dots, \nu_d) \sim \frac{1}{(p-1)s^d}\binom{p+d-1}{d}^k \quad \textit{as} \quad k \to \infty,$$

and hence

$$\lim_{k\to\infty} \frac{\log f^{(k)}(\rho, \nu_1, \dots, \nu_d)}{\log p^k} = \frac{\log \binom{p+d-1}{d}}{\log p}.$$

For $\rho = 0$ we have that

$$\sum_{\nu_1,\dots,\nu_d} f^{(k)}(\rho,\nu_1,\dots,\nu_d) = p^{dk} - \binom{p+d-1}{d}^k = p^{dk}\left(1 - \frac{1}{d!}\prod_{j=1}^{d}\frac{p+j-1}{p}\right)$$

is the number of zero blocks in $\mathbb{C}^{(k)}$

Proof. The first part of the theorem follows from the previous lemmas by applying Perron-Frobenius theory. The last statement follows from another part of Dickson's theorem [3], [5, p. 273] which generalizes Kummer's theorem [17]. This result states that the exponent of the highest power of p that divides the multinomial coefficient $M(n_1,\dots,n_d)$ is the same as the number of carries that occur when $n_1,\dots,n_d$ are expressed in radix p— $n_i = \sum_{j=0}^{k-1} n_{ij}p^j$—and added. This number will be zero precisely when each sum $\sum_{i=1}^{d} n_{ij} < p$, and the number of d-tuples $(n_1,\dots,n_d)$ for which this occurs is $\binom{p+d-1}{d}^k$. Therefore the number of nonzero blocks $M(n_1',\dots,n_d')H_{n_1'',\dots,n_d''}$ in $\mathbb{C}^k$ is $\binom{p+d-1}{d}^k$. □

Corollary 2. *Let*

$$R_k(\rho) = \#\{(n_1,\dots,n_d) : 0 \le n_1,\dots,n_d < rp^k, C(n_1,\dots,n_d) \equiv \rho \pmod p\},$$

the number of $C(n_1,\dots,n_d)$'s in the initial $rp^k \times \cdots \times rp^k$ block congruent to ρ modulo p. Then the asymptotic density of residue ρ, where $1 \le \rho < p$, is

$$R_k(\rho) \sim \frac{1}{(p-1)}\binom{r+d-1}{d}\binom{p+d-1}{d}^k \quad \text{as } k \to \infty,$$

and so the logarithmic density, or box-counting dimension, of the set of generalized binomial coefficients that are congruent to ρ is

$$\lim_{k\to\infty}\frac{\log R_k(\rho)}{\log(rp^k)} = \frac{\binom{p+d-1}{d}}{\log p}.$$

Proof. Let

$$g(\rho,\nu_1,\dots,\nu_d) = \#\{(i_1,\dots,i_d)) : 0 \le i_1,\dots,i_d < r, H_{\nu_1,\dots,\nu_d}(i_1,\dots,i_d) \equiv \rho \pmod p\},$$

the number of entries in the $r\times\cdots\times r$ array $H_{\nu_1,\dots,\nu_d}$ that are congruent to ρ modulo p. Then

$$R_k(\rho) = \sum_{1\le\tilde\rho<p}\ \sum_{0\le\nu_1,\dots,\nu_d<s} f^{(k)}(\rho\tilde\rho^{-1},\nu_1,\dots,\nu_d)g(\tilde\rho,\nu_1,\dots,\nu_d),$$

so

$$\begin{aligned}
R_k(\rho) &\sim \sum_{1\le\tilde\rho<p}\ \sum_{0\le\nu_1,\dots,\nu_d<s} \frac{\binom{p+d-1}{d}^k}{(p-1)s^d} g(\tilde\rho,\mu,\nu) \\
&= \frac{\binom{p+d-1}{d}^k}{(p-1)s^d} \sum_{0\le\nu_1,\dots,\nu_d<s}\ \sum_{1\le\tilde\rho<p} g(\tilde\rho,\nu_1,\dots,\nu_d) \\
&= \frac{\binom{p+d-1}{d}^k}{(p-1)s^d}\cdot s^d\cdot\binom{r+d-1}{d}. \quad \square
\end{aligned}$$

HAUSDORFF DIMENSION OF $C(n_1,\dots,n_d) \mod p$

A "fractal set" corresponding to the pattern of all nonzero residues of the generalized binomial coefficients modulo a prime p is constructed as a subset of the square $[0,1)\times[0,1)$ by "tremas" as follows. Flath and Peele [7] give an alternative, rescaled lattice construction.

While this report is still under construction, the Hausdorff dimension discussion is carried out in the case $d=2$, but it should be clear that the results of the previous sections may be employed in generalizing the result to arbitrary $d\ge 2$.

For each k let $\mathcal{G}_k$ denote the class of sets

$$G^{(k)}_{m,n} = \bigcup_{\substack{0\le i,j<r\\ i+j<r}} \left[\frac{mr+i}{rp^k},\frac{mr+i+1}{rp^k}\right)\times\left[\frac{nr+j}{rp^k},\frac{nr+j+1}{rp^k}\right)$$

with $0\le m,n<p^k$ and $p\nmid B(m,n)$, and let G_k be their union. Theorem 1 and Lucas's theorem imply that $G^{(k)}_{m,n}$ is contained in some set in $\mathcal{G}_{k-1}$ and contains a finite number of disjoint sets of $\mathcal{G}_{k+1}$, and $G_{k+1}\subset G_k$. Accordingly our fractal set is

$$G := \bigcap_{k\in\mathbb{N}} G_k.$$

Theorem 3. *If p is a prime that does not divide b, then the fractal set G constructed above has Hausdorff dimension*

$$\dim_H(G) = \frac{\log\binom{p+1}{2}}{\log p}.$$

Proof. The proof uses (1) the fact that

$$\dim_H G\le \dim_B G,$$

where $\dim_B G$ is the box-counting dimension of G, and (2) the mass distribution principle [6, p. 55]: if μ is a measure on a set F and for some s there are numbers $c > 0, \delta > 0$ such that

$$\mu(U) \leq c|U|^s$$

for all sets U with $|U| \leq \delta$ (where $|U|$ is the diameter of U), then the Hausdorff measure $\mathcal{H}^s(F) \geq \mu(F)/c$ and $s \leq \dim_H(F)$. The box-counting dimension of G may be calculated [6, p. 41] by the formula

$$\dim_B G = \lim_{k\to\infty} \frac{\log N_{\delta_k}(G)}{-\log \delta_k},$$

where $N_\delta(G)$ is the smallest number of δ-mesh squares that intersect the set G, provided that the sequence (δ_k) decreases to zero and $\delta_{k+1} \geq \eta\delta_k$ for some positive constant η. Let us choose $\delta_k = 1/(rp^k)$ (and $\eta = 1/p$). Then

$$N_{\delta_k}(G) = \frac{r(r+1)}{2}\left[\frac{p(p+1)}{2}\right]^k,$$

so

$$\begin{aligned}
\dim_B(G) &= \lim_{k\to\infty} \frac{\log N_{\delta_k}(G)}{-\log \delta_k} \\
&= \lim_{k\to\infty} \frac{\log\left(\frac{r(r+1)}{2}\left[\frac{p(p+1)}{2}\right]^k\right)}{-\log[1/(rp^k)]} \\
&= \lim_{k\to\infty} \frac{\log\frac{r(r+1)}{2} + k\log\frac{p(p+1)}{2}}{\log r + k\log p} \\
&= \frac{\log\binom{p+1}{2}}{\log p}.
\end{aligned}$$

Now let μ be the "natural measure" defined by repeated subdivision [6, pp. 13–14] that assigns weight $\binom{p+1}{2}^{-k}$ to each set in $\mathcal{G}_k$ and weight 0 to the complement of G_k. (At the next stage, the weight of each $G_{m,n}^{(k)}$ is evenly divided among the $\binom{p+1}{2}$ sets in $\mathcal{G}_{k+1}$ contained therein.) We shall see that there exist $c > 0$ and $\delta > 0$ such that

$$\mu(U) \leq c|U|^d \quad \text{where} \quad d := \frac{\log\binom{p+1}{2}}{\log p}$$

for all sets U with diameter $|U| \leq \delta$. Let $\delta \in (0,1)$. Suppose $|U| \leq \delta$. Let k be the integer such that $1/p^{k+1} \leq |U| < 1/p^k$. Note that then $1/p^k \leq p|U|$ and U meets at most four of the sets in $\mathcal{G}_k$ (because U is contained in a square of side $|U|$ with sides parallel the coordinate axes, and this containing square can intersect no more than four $G_{m,n}^{(k)}$'s).

Therefore,

$$\begin{aligned}\mu(U) &\leq 4\frac{1}{\binom{p+1}{2}^k} \\ &= \frac{4}{(p^d)^k} \quad \left[\text{because } d = \frac{\log\binom{p+1}{2}}{\log p}\ \right] \\ &= 4\left(\frac{1}{p^k}\right)^d \\ &\leq 4(p|U|)^d,\end{aligned}$$

so $\mu(U) \leq c|U|^d$ for all sets U with $|U| \leq \delta$ where $c = 4p^d$. By the mass distribution principle, $d \leq \dim_H G$. But from before, $\dim_B G = d$, and we know $\dim_H G \leq \dim_B G$, so we must have $\dim_H G = d = \log\binom{p+1}{2}/\log p$. $\square$

The generalization to arbitrary $d \geq 2$ is as follows.

Theorem 4. *Let* $s_k = rp^k$.
The Hausdorff dimension of $\cap_k \cup \{[n_1/s_k, (n_1+1)/s_k) \times \cdots \times [n_d/s_k, (n_d+1)/s_k) | 0 \leq n_1, \ldots, n_d < s_k, p \nmid C(n_1, \ldots, n_d)\}$ *is* $\log\binom{p+d-1}{d}/\log p$.

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