PROPERTIES OF O-REGULARLY VARYING SEQUENCES: ELEMENTARY PROOFS

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ABSTRACT. Two sequences $f, g : \mathbb{N} \to \mathbb{R}^+$ are "asymptotically of the same order"—written $f(n) \asymp g(n)$—if $f(n) = O(g(n))$ and $g(n) = O(f(n))$, and f is "O-regularly varying"—written $f \in OR_{seq}$—if $f(\lfloor \lambda n \rfloor) \asymp f(n)$ for all positive λ. This paper provides proofs of major theorems about such sequences that are accessible to undergraduates. These theorems include a representation formula ($f \in OR_{seq}$ iff $\exists$ bounded sequences g, h such that $f(n) = \exp\{g(n) + \sum_{k=1}^{n} h(k)/k\}$), a characterization of OR_{seq} analogous to the Bojanic-Seneta characterization of (ordinary) regularly varying sequences, and a discrete version of the OR Karamata Tauberian theorem/de Haan-Stadtmüller theorem. One corollary states that for f nondecreasing, $f(1) + \cdots + f(n) \asymp nf(n)$ iff $f(2n)/f(n)$ is bounded.

1. INTRODUCTION

One purpose of this paper is to provide an introduction to the theory of regular variation and O-regular variation that is accessible to undergraduate mathematics majors who have not yet learned about measure theory or meager sets. Another purpose is to provide, to the extent possible, a self-contained exposition of the theory of O-regular variation in the discrete case, i.e., for O-regularly varying sequences. Accordingly, this paper is a series of exercises in what Graham, Knuth, and Patashnik [6, p. vi] call "concrete mathematics," a practical blend of CONtinuous and disCRETE mathematics. The student reader is invited to participate in the development here presented. In most cases, the numbered formulas should be viewed as assertions to be verified. (Usually these are "quickies.") The reader is also invited to provide proofs, which vary in difficulty, of other assertions in the text.

The passages outlining the wider context of the the theory of regular variation are written at the instructor level, but should still be somewhat accessible to the student.

2. ASYMPTOTIC EQUIVALENCE AND REGULAR VARIATION

The student of calculus who sets up a Riemann sum for $\int_0^1 x^\rho \, dx$ naturally encounters a sum of the form

$$\sum_{k=1}^{n} k^\rho.$$

Evaluation of this sum for various values of ρ leads to formulas like

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}.$$

Several examples of this sort will lead the attentive student to the conjecture that

$$\sum_{k=1}^{n} k^\rho = \frac{n^{\rho+1}}{\rho+1} + \text{ lower order terms},$$

or

$$\sum_{k=1}^{n} k^\rho \sim \frac{n^{\rho+1}}{\rho+1},$$

where

$$f(n) \sim g(n) \text{ (“}f\text{ is asymptotic to }g\text{”) if and only if } \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

In fact, the conjecture is true for $\rho > -1$ and $n \in \mathbb{N}$. As a generalization of a sequence of powers (n^ρ), in 1930 Jovan Karamata introduced the notion of a regularly varying sequence. In his original definition [8], a sequence of positive numbers (c(n)) is *regularly varying* if, for some $\rho \in (-1, \infty)$,

$$\sum_{k=1}^{n} c(k) \sim \frac{nc(n)}{\rho+1}. \tag{1}$$

We shall write c or $c(n) \in R_\rho$ if c is a regularly varying sequence with “index” ρ. By 1933 he had adopted essentually the now-standard definition of a regularly varying function: A positive function f defined on some ray $[X, \infty)$ is *regularly varying* if

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \psi(\lambda) \qquad (\lambda > 0)$$

for some function ψ. It is a theorem of the theory that $\psi(\lambda) = \lambda^\rho$ for some finite ρ. And it is natural, then, to define a sequence $f : \mathbb{N} := \{1, 2, 3, \dots\} \to \mathbb{R}^+ := (0, \infty)$ to be *regularly varying* if

$$\lim_{n\to\infty} \frac{f(\lfloor \lambda n \rfloor)}{f(n)} = \psi(\lambda) \qquad (\lambda > 0).$$

This matter has been scrutinized by Bojanic and Seneta [2]. The study of regularly varying functions and their generalizations has an extensive literature in real analysis, now masterfully incorporated into the *Encyclopedia of Mathematics* by Bingham, Goldie, and Teugels [1].

3. When Is $f(1) + \cdots + f(n) \asymp nf(n)$?

Two sequences of positive numbers $(f(n))_{n=1}^{\infty}$ and $(g(n))_{n=1}^{\infty}$ are asymptotically of the same order, or have the same rate of growth—written $f(n) \asymp g(n)$ or $f(n) = \Theta(g(n))$—if

$$f(n) = O(g(n)) \quad \text{and} \quad g(n) = O(f(n)) \quad \text{as} \quad n \to \infty,$$

i.e., there exist constants c_1 and c_2 and an index n_0 such that

$$0 < c_1 \leq f(n)/g(n) \leq c_2 < \infty \quad \text{for} \quad n \geq n_0.$$

It is easy to show that for positive sequences this is equivalent to the existence of constants C_1 and C_2 such that

$$0 < C_1 \leq f(n)/g(n) \leq C_2 < \infty \quad \text{for } \textit{every } n \in \mathbb{N}.$$

A student frequently encounters sequences $(f(n))$ of positive numbers for which the sequence of partial sums $f(1) + \cdots + f(n)$ is asymptotically of the same order as $(nf(n))$, or, in other words, for which the sequence of arithmetic means $(\{f(1) + \cdots + f(n)\}/n)$ is asymptotically of the same order as the original sequence.

For example, a student of computer science, in considering algorithms for sorting arbitrary lists of n items, may learn that information theory tells us that this task has complexity $\log_2 n!$, and then it follows that

$$\log_2 n! = \sum_{k=1}^{n} \log_2 k = \int_1^n \log_2 x \, dx + O(\ln n) = n \log_2 n - n/\ln 2 + O(\ln n),$$

so that

$$\sum_{k=1}^{n} \log_2 k \asymp n \log_2 n.$$

Polynomial sequences all exhibit this behavior. For example, as we have seen, if $\rho \in \mathbb{N}$

$$\sum_{k=1}^{n} k^{\rho} \sim \frac{1}{\rho+1} \cdot n \cdot n^{\rho}.$$

whence

$$\sum_{k=1}^{n} k^{\rho} \asymp n \cdot n^{\rho}.$$

Divergent p-series also exhibit this behavior for $p < 1$ (but not for $p = 1$):

$$\sum_{k=1}^{n} k^{-p} \sim \int_1^n x^{-p} \, dx \sim \frac{n^{-p+1}}{-p+1} \asymp n \cdot n^{-p}.$$

Fluctuating sequences $(f(n))$ may also exhibit this behavior. For example, if $f(n) = 2 + \sin n$, then $1 \leq f(n) \leq 3$, so that

$$n \leq \sum_{k=1}^{n} f(k) \leq 3n,$$

whence

$$\frac{1}{3} \le \frac{\sum_{k=1}^{n} f(k)}{nf(n)} \le 3,$$

and so $f(1)+\cdots+f(n) \asymp nf(n)$. More generally, if $f(n) = n^{-p}(2+\sin n)$ where $p < 1$, then we still have $f(1)+\cdots+f(n) \asymp nf(n)$, but **NOTE:** we do **not** have $f(1)+\cdots+f(n) \sim cnf(n)$ for any constant c.

On the other hand, convergent series of positive numbers never exhibit this behavior. For if $f(1)+\cdots+f(n) \le Cnf(n)$ for some $C > 0$, then $f(n) \ge f(1)/Cn$, whence $\sum f(n)$ must diverge.

Also, sequences that increase geometrically do not exhibit this behavior: If $r > 1$, then

$$\sum_{k=1}^{n} r^k = \frac{r^{n+1}-1}{r-1},$$

so

$$\frac{\sum_{k=1}^{n} r^k}{nr^n} = \frac{r-r^{-n}}{n(r-1)},$$

and as $n \to \infty$, this approaches 0, so it is not bounded away from 0.

So, when is $f(1)+\cdots f(n) \asymp nf(n)$?

A complete answer to this question turns out to involve the O-regularly varying sequences. In line with Karamata's original definition (1) of a regularly varying sequence, one might be tempted to *define* a O-regularly varying sequence $f : \mathbb{N} \to \mathbb{R}^+$ by the condition

$$\sum_{k=1}^{n} f(k) \asymp nf(n).$$

But just as (1) was restricted to $\rho > -1$, this proposed definition turns out to be too restrictive. The standard definition of a O-regularly varying *function* is the following.

Definition 1. A function $f : [X, \infty) \to \mathbb{R}^+$ is *O-regularly varying* if

$$0 < \liminf_{x\to\infty} \frac{f(\lambda x)}{f(x)} \le \limsup_{x\to\infty} \frac{f(\lambda x)}{f(x)} < \infty.$$

Accordingly, we shall define a O-regularly varying *sequence* as follows.

Definition 2. A sequence $f : \mathbb{N} \to \mathbb{R}^+$ is *O-regularly varying* if

$$0 < f_*(\lambda) := \liminf_{n\to\infty} \frac{f(\lfloor \lambda n \rfloor)}{f(n)} \le f^*(\lambda) := \limsup_{n\to\infty} \frac{f(\lfloor \lambda n \rfloor)}{f(n)} < \infty,$$

or, equivalently, $f(\lfloor \lambda n \rfloor) \asymp f(n)$. The set of all O-regularly varying sequences is denoted OR_{seq}.

In the next sections we shall see that only O-regularly varying sequences satisfy $f(1)+\cdots f(n) \asymp nf(n)$, and we shall determine precisely which ones do so.

4. Increasing Sequences

Theorem 1. *Assume that f is a positive, nondecreasing sequence. Then $f(1)+\cdots+f(n) \asymp nf(n)$ if and only if the sequence $(f(2n)/f(n))$ is bounded, or, equivalently, $f^*(2) < \infty$.*

Proof. For a positive, nondecreasing f we always have

$$\frac{f(1)+\cdots+f(n)}{nf(n)} \leq 1, \tag{2}$$

so we need only show that there exists a constant $C > 0$ such that

$$\frac{f(1)+\cdots+f(n)}{nf(n)} \geq C \tag{3}$$

for all $n \geq 1$ if and only if there exists a constant B such that $f(2n)/f(n) \leq B$ for all $n \geq 1$. First assume $f(2n)/f(n) \leq B$ for $n \geq 1$. Write $n = 2m$ or $n = 2m+1$; then

$$\begin{aligned}
\frac{f(1)+\cdots+f(n)}{nf(n)} &\geq \frac{\overbrace{f(m+1)+\cdots+f(n)}^{m \text{ or } m+1 \text{ terms}}}{(2m \text{ or } 2m+1)f(n)} && (4)\\
&\geq \frac{(m \text{ or } m+1)f(m+1)}{[2m \text{ or } 2(m+1)]f(2(m+1))} && (5)\\
&\geq \frac{1}{2}\cdot\frac{1}{B} =: C > 0. && (6)
\end{aligned}$$

For the converse, assume (3). Choose integers p and q such that $p > q \geq 1$ and $1 - q/p < C^2$. Then, using (3) and (2) and monotonicity of f, check that

$$\begin{aligned}
\frac{C}{1} \leq \frac{\frac{f(1)+\cdots+f(pn)}{pnf(pn)}}{\frac{f(1)+\cdots+f(qn)}{qnf(qn)}} &= \frac{qf(qn)}{pf(pn)}\left\{1 + \frac{f(qn+1)+\cdots+f(pn)}{f(1)+\cdots+f(qn)}\right\} && (7)\\
\leq \frac{qf(qn)}{pf(pn)}\left\{1 + \frac{(pn-qn)f(pn)}{Cqnf(qn)}\right\} &= \frac{qf(qn)}{pf(pn)} + \frac{p-q}{pC}, && (8)
\end{aligned}$$

where $(p-q)/pC < C$. Therefore,

$$\frac{qf(qn)}{pf(pn)} \geq C - \frac{p-q}{pC} =: D > 0, \tag{9}$$

so

$$\frac{f(pn)}{f(qn)} \leq \frac{q}{pD} =: E < \infty \quad \text{and} \quad f(pn) \leq Ef(qn). \tag{10}$$

If $f(p^{k-1}n) \leq E^{k-1}f(qn)$ for all n, then check that

$$f(p^k n) = E^k f(qn) \tag{11}$$

for all n. By induction,

$$f(p^k n) \leq E^k f(qn) \text{ for all } k \text{ and } n.$$

Choose integer $k \geq \log_p 4q$. If $n \geq q$, then $n = mq + r$ with $m \geq 1$ and $0 \leq r < q$ and

$$\frac{f(2n)}{f(n)} = \frac{f(2(mq+r))}{f(mq+r)} \leq \frac{f(2(m+1)q)}{f(mq)} \leq \frac{f(4mq)}{f(mq)} \leq \frac{f(p^k m)}{f(qm)} \leq E^k. \tag{12}$$

Thus, $f(2n)/f(n)$ is bounded by B defined as the maximum of E^k and the values of $f(2n)/f(n)$ for $n = 1, \ldots, q-1$. □

A related result, proved long ago by Feller [5], [1, p. 65], is the following.

Theorem 2. *If f is a positive, nondecreasing function, then finiteness of $f^*(\lambda_0)$ at one point $\lambda_0 > 1$ implies that f is O-regularly varying.*

5. Positive Sequences in General

Theorem 3. *Let $f : \mathbb{N} \to \mathbb{R}^+$. The following are equivalent:*

(1) *$f(1) + \cdots + f(n) \asymp nf(n)$;*
(2) *there exist sequences $(g(n))$ and $(h(n))$ bounded away from 0 and ∞ (or, $g(n) \asymp 1$ and $h(n) \asymp 1$) such that*

$$f(n) = \frac{g(n)}{n} \exp \sum_{k=1}^{n} \frac{h(k)}{k};$$

(3) *there exists $\alpha > -1$ such that $n^{-\alpha} f(n)$ is "almost decreasing," (i.e., there exist $M > 0$ and $n_0 \in \mathbb{N}$ such that $n^{-\alpha} f(n) \leq M k^{-\alpha} f(k)$ whenever $n_0 \leq k \leq n$), and there exists $\beta > -1$ such that $n^{-\beta} f(n)$ is "almost increasing," (i.e., there exist $m > 0$ and $n_0' \in \mathbb{N}$ such that $mk^{-\beta} f(k) \leq n^{-\beta} f(n)$ whenever $n_0' \leq k \leq n$).*

The proof is given via the following five lemmas showing that

Statement 1 ⇒ Statement 2 ⇒ Statement 3 ⇒ Statement 1.

Let $f : \mathbb{N} \to \mathbb{R}^+$, and define $b : \mathbb{N} \cup \{0\} \to \mathbb{R}^+$ as follows:

$$b(n) = \begin{cases} 1 \text{ if } n = 0; \\ (n+1)f(n+1)/\{f(1) + \cdots + f(n)\} \text{ if } n \in \mathbb{N}. \end{cases} \tag{13}$$

Lemma 1. *For $n \in \mathbb{N}$,*

$$\sum_{k=1}^{n} f(k) = f(1) \exp \sum_{k=1}^{n-1} \ln \left\{1 + \frac{b(k)}{k+1}\right\}$$

and

$$f(n) = f(1) \frac{b(n-1)}{n} \exp \sum_{k=1}^{n-2} \ln \left\{1 + \frac{b(k)}{k+1}\right\}. \tag{14}$$

Proof.

$$1 + \frac{b(k)}{k+1} = 1 + \frac{f(k+1)}{f(1) + \cdots + f(k)} = \frac{f(1) + \cdots + f(k+1)}{f(1) + \cdots + f(k)}. \tag{15}$$

$$\ln\left\{1+\frac{b(k)}{k+1}\right\} = \ln\{f(1)+\cdots+f(k+1)\} - \ln\{f(1)+\cdots+f(k)\}. \tag{16}$$

$$\sum_{k=1}^{n}\ln\left\{1+\frac{b(k)}{k+1}\right\} = \text{ telescoping sum} = \ln\{f(1)+\cdots+f(n+1)\} - \ln f(1). \tag{17}$$

$$\exp\sum_{k=1}^{n}\ln\left\{1+\frac{b(k)}{k+1}\right\} = \frac{f(1)+\cdots+f(n+1)}{f(1)} \quad \text{for } n \geq 0. \tag{18}$$

The first conclusion now follows. By the definition of $b(\cdot)$ and the first conclusion, we have, for $n \geq 1$,

$$f(n+1) = \frac{b(n)}{n+1}\{f(1)+\cdots+f(n)\} = \frac{b(n)}{n+1}f(1)\exp\sum_{k=1}^{n-1}\ln\left\{1+\frac{b(k)}{k+1}\right\}, \tag{19}$$

and the first and last terms are also equal when $n = 0$. □

Lemma 2. *If* $\sum_{k=1}^{n} f(k) \asymp nf(n)$, *then* $b(n) \asymp 1$.

Proof. Assume

$$0 < C_1 \leq \frac{f(1)+\cdots f(n)}{nf(n)} \leq C_2 < \infty.$$

for $n \in \mathbb{N}$. Then

$$C_1 \leq \frac{\sum_{k=1}^{n+1} f(k)}{(n+1)f(n+1)} = \frac{\sum_{k=1}^{n} f(k)}{(n+1)f(n+1)} + \frac{1}{n+1} \leq C_2, \tag{20}$$

so

$$C_1 - \frac{1}{n+1} \leq \frac{1}{b(n)} \leq C_2 - \frac{1}{n+1} < C_2. \tag{21}$$

For sufficiently large n—$n \geq n_0$, say— $C_1 - 1/(n+1) \geq C_1 - 1/(n_0+1) > 0$, so that for $n \geq n_0$,

$$\infty > \frac{1}{C_1 - 1/(n_0+1)} \geq b(n) > \frac{1}{C_2} > 0. \tag{22}$$

Thus $b(n) \asymp 1$. □

Lemma 3. *If* $b(n) \asymp 1$, *then there exist sequences* $(g(n))$ *and* $(h(n))$ *such that* $g(n) \asymp 1$, $h(n) \asymp 1$, *and*

$$f(n) = \frac{g(n)}{n}\exp\sum_{k=1}^{n}\frac{h(k)}{k} \quad \textit{for } n \in \mathbb{N}. \tag{23}$$

Proof. For $n \geq 3$ let

$$h(n) = n \ln \left\{ 1 + \frac{b(n-2)}{n-1} \right\}$$

and let $h(1)$ and $h(2)$ be arbitrary positive numbers. For $n \in \mathbb{N}$ let

$$g(n) = f(1) b(n-1) \exp\{-h(1) - h(2)/2\}.$$

Then a simple calculation (exercise) shows that Equation (14) implies Equation (23). Now suppose that $0 < b_l \leq b(n) \leq b_u < \infty$ for $n \geq 0$. Basic calculus (exercise) shows that $\ln(1+x)$ is increasing and $x - x^2/2 \leq \ln(1+x) \leq x$ for $x \geq 0$. Therefore, for $n \geq 3$,

$$h(n) = n \ln \left\{ 1 + \frac{b(n-2)}{n-1} \right\} \leq n \ln \left\{ 1 + \frac{b_u}{n-1} \right\} \leq n \frac{b_u}{n-1} < 2b_u, \tag{24}$$

and

$$h(n) \geq n \ln \left\{ 1 + \frac{b_l}{n-1} \right\} \geq n \frac{b_l}{n-1} \left(1 - \frac{1}{2} \frac{b_l}{n-1} \right) > b_l \left(1 - \frac{b_l}{2} \right) > 0, \tag{25}$$

since $0 < b_l \leq b(0) = 1$, so that for $n \geq 1$ we have

$$0 < \min\{h(1), h(2), b_l(1 - b_l/2)\} \leq h(n) \leq \max\{h(1), h(2), 2b_u\}. \tag{26}$$

Also for $n \geq 1$ we have

$$0 < f(1) b_l e^{-h(1) - h(2)/2} \leq g(n) \leq f(1) b_u e^{-h(1) - h(2)/2} < \infty. \quad \square \tag{27}$$

Lemma 4. *If $f(n)$ is given by (23) where $g(n) \asymp 1$ and $h(n) \asymp 1$, then there exist constants α and β, both greater than -1, such that $n^{-\alpha} f(n)$ is almost decreasing and $n^{-\beta} f(n)$ is almost increasing. If $0 < h_l \leq h(n) \leq h_u < \infty$ for all n, then we may take $\alpha = h_u - 1$ and $\beta = h_l - 1$.*

Proof. We shall show that there exist constants α and β and positive constants m and M such that

$$n^{-\alpha} f(n) \leq M k^{-\alpha} f(k)$$

and

$$m k^{-\beta} f(k) \leq n^{-\beta} f(n)$$

for $1 \leq k \leq n < \infty$. Alternatively, these inequalities may be written in the form we shall call "Potter-type bounds" [1, pp. 25,72]:

$$m \left(\frac{n}{k} \right)^{\beta} \leq \frac{f(n)}{f(k)} \leq M \left(\frac{n}{k} \right)^{\alpha} \qquad (n \geq k \geq 1). \tag{28}$$

Assume that $0 < g_l \leq g(n) \leq g_u < \infty$ and $0 < h_l \leq h(n) \leq h_u < \infty$ for $n \geq 1$. Assume $1 \leq k \leq n$. By hypothesis,

$$\frac{f(n)}{f(k)} = \frac{g(n)/n}{g(k)/k} \cdot \frac{\exp \sum_{j=1}^{n} h(j)/j}{\exp \sum_{j=1}^{k} h(j)/j} = \frac{k g(n)}{n g(k)} \exp \sum_{j=k+1}^{n} \frac{h(j)}{j}, \tag{29}$$

where

$$0 < \frac{g_l}{g_u} \le \frac{g(n)}{g(k)} \le \frac{g_u}{g_l} < \infty \tag{30}$$

and

$$h_l \int_{k+1}^{n+1} \frac{1}{x}\,dx \le h_l \sum_{j=k+1}^{n} \frac{1}{j} \le \sum_{j=k+1}^{n} \frac{h(j)}{j} \le h_u \sum_{j=k+1}^{n} \frac{1}{j} \le h_u \int_{k}^{n} \frac{1}{x}\,dx. \tag{31}$$

Then

$$h_l \ln \frac{n+1}{k+1} \le \sum_{j+k+1}^{n} \frac{h(j)}{j} \le h_u \ln \frac{n}{k}, \tag{32}$$

whence

$$\left(\frac{n}{2k}\right)^{h_l} < \left(\frac{n+1}{k+1}\right)^{h_l} \le \exp \sum_{j=k+1}^{n} \frac{h(j)}{j} \le \left(\frac{n}{k}\right)^{h_u}. \tag{33}$$

Take $m = 2^{-h_l} g_l/g_u$ and $M = g_u/g_l$; then

$$m \left(\frac{n}{k}\right)^{h_l - 1} \le \frac{f(n)}{f(k)} \le M \left(\frac{n}{k}\right)^{h_u - 1}. \tag{34}$$

Let $\alpha = h_u - 1$ and $\beta = h_l - 1$; then

$$mk^{-\beta} f(k) \le n^{-\beta} f(n) \quad \text{and} \quad n^{-\alpha} f(n) \le Mk^{-\alpha} f(k). \quad \square \tag{35}$$

Lemma 5. *If there exist constants α and β greater than -1 such that $n^{-\alpha} f(n)$ is almost decreasing and $n^{-\beta} f(n)$ is almost increasing, then $f(1) + \cdots + f(n) \asymp nf(n)$.*

Proof. The hypothesis implies that there exist positive constants m and M and indices n_0 and n_0' such that $n^{-\alpha} f(n) \le Mk^{-\alpha} f(k)$ for $n_0 \le k \le n$ and $mk^{-\beta} f(k) \le n^{-\beta} f(n)$ for $n_0' \le k \le n$. Then for $n \ge k \ge n_0'' := \max\{n_0, n_0'\}$,

$$\frac{k^\alpha}{Mn^\alpha} \le \frac{f(k)}{f(n)} \le \frac{k^\beta}{mn^\beta} \tag{36}$$

and $nf(n) \ge m(n_0'')^{-\beta} f(n_0'') n^{1+\beta} = cn^{1+\beta}$ (say), where $1 + \beta > 0$. Then

$$\frac{\sum_{k=n_0''}^{n} k^\alpha}{Mn^{1+\alpha}} \le \frac{\sum_{k=n_0''}^{n} f(k)}{nf(n)} \le \frac{\sum_{k=n_0''}^{n} k^\beta}{mn^{1+\beta}} \tag{37}$$

and

$$0 < \frac{f(1) + \cdots + f(n_0'' - 1)}{nf(n)} \le \frac{f(1) + \cdots + f(n_0'' - 1)}{cn^{1+\beta}} \le \frac{f(1) + \cdots + f(n_0'' - 1)}{c}. \tag{38}$$

Now

$$\sum_{k=n_0''}^{n} k^\alpha \sim \int_{n_0''}^{n} x^\alpha\,dx \sim \frac{n^{1+\alpha}}{1+\alpha} - \frac{(n_0'')^{1+\alpha}}{1+\alpha} \sim \frac{n^{1+\alpha}}{1+\alpha} \tag{39}$$

and similarly for $\sum_{k=n_0''}^{n} k^\beta$, since both α and β are greater than -1. Therefore, there are constants C_1 and C_2 such that

$$0 < C_1 \leq \frac{f(1) + \cdots + f(n_0'' - 1) + f(n_0'') + \cdots + f(n)}{nf(n)} \leq C_2 < \infty \tag{40}$$

for all sufficiently large n. □

Corollary 1. *Assume $f : \mathbb{N} \to \mathbb{R}^+$ is nondecreasing and $\{f(2n)/f(n)|n \in \mathbb{N}\}$ is bounded (above). Then f is a O-regularly varying sequence.*

Proof. Exercise.

Corollary 2. *Let $f : \mathbb{N} \to \mathbb{R}^+$. Then $f \in OR_{seq}$ if and only if f satisfies Potter-type bounds (28) for some α, β, m, M.*

Proof. Exercise.

Corollary 3. *Let $f : \mathbb{N} \to \mathbb{R}^+$. Then $f(1) + \cdots + f(n) \asymp nf(n)$ if and only if there exist constants $\alpha > -1$, $\beta > -1$, $m > 0$, and $M > 0$ such that*

$$m\lambda^\beta \leq f_*(\lambda) \leq f^*(\lambda) \leq M\lambda^\alpha$$

for every real $\lambda \geq 1$.

Proof. Use parts 1 and 3 of Theorem 3 ....

Corollary 4. *Let $f : \mathbb{N} \to \mathbb{R}^+$. Let σ be a real number. Then*

$$\sum_{k=1}^{n} k^\sigma f(k) \asymp n^{\sigma+1} f(n)$$

if and only if there exist bounded sequences g and h such that

$$f(n) = \exp\{g(n) + \sum_{k=1}^{n} h(k)/k\}$$

where

$$h_L := \liminf_{n\to\infty} h(n) > -(\sigma + 1).$$

Proof. Exercise.

Corollary 5. *Let $f : \mathbb{N} \to \mathbb{R}^+$. Let ρ be a real number. Then f is regularly varying with index ρ if and only if there exist bounded sequences g and h such that*

$$f(n) = \exp\{g(n) + \sum_{k=1}^{n} h(k)/k\}$$

where $\lim_{n\to\infty} g(n)$ exists and $\lim_{n\to\infty} h(n) = \rho$.

Proof. Exercise.

6. O-Regularly Varying Functions

A real analyst may regard the theory of regularly varying and O-regularly varying sequences as simple special cases of the corresponding theory for functions of a real variable. As an illustration of the connection, we may use Theorem 3 to show that the function $g(x) := f(\lfloor x \rfloor)$ is a O-regularly varying function. It takes a bit more work to show the expected result that if $g : [1, \infty) \to \mathbb{R}^+$ is a O-regularly varying function, then $(g(n))_{n=1}^{\infty}$ is a O-regularly varying sequence.

Essentially all the results stated here are counterparts of theorems for O-regularly varying functions. For example, the representation formula in part 2 of Theorem 3 almost has its parallel in the representation formula for O-regularly varying functions f:

$$f(x) = \exp\left\{\eta(x) + \int_1^x \frac{\xi(t)}{t}\,dt\right\},$$

where η and ξ are bounded and measurable functions. The completely parallel discrete result is the following.

Theorem 4. *Let $f : \mathbb{N} \to \mathbb{R}^+$. Then f is O-regularly varying if and only if there exist bounded sequences g, h such that*

$$f(n) = \exp\{g(n) + \sum_{k=1}^{n} h(k)/k\}. \tag{41}$$

In order to relate the representation formula (41) to the representation formula in Theorem 3, let us first rewrite the latter as

$$f(n) = \frac{G(n)}{n} \exp\{\sum_{k=1}^{n} H(k)/k\}.$$

Then show that

$$f(n) = \exp\{g(n) + \sum_{k=1}^{n} h(k)/k\} \tag{42}$$

where

$$g(n) := \ln G(n) + \sum_{k=1}^{n} \frac{1}{k} - \ln n$$

and

$$h(n) := H(n) - 1,$$

and note that positive $G(n) \asymp 1$ if and only if $g(n)$ is bounded, and $\liminf_{n\to\infty} H(n) > 0$ if and only if $\liminf_{n\to\infty} h(n) > -1$. Thus, $f(1) + \cdots + f(n) \asymp nf(n)$ if and only if f has the representation (41) with g and h bounded and $\liminf_{n\to\infty} h(n) > -1$.

The proof of the "if" part of Theorem 4 is a good, elementary exercise. The proof of the "only if" part may be another matter. It may be deduced from the real-variable theory, or the real-variable theory proof may be adapted to the discrete context. Then one goes through the Uniform Convergence Theorem, the discrete version of which is as follows.

Theorem 5. *Let $f : \mathbb{N} \to \mathbb{R}^+$. If f is O-regularly varying, i.e., for all $\lambda > 1$,*

$$0 < \liminf_{n\to\infty} \frac{f(\lfloor \lambda n \rfloor)}{f(n)} \leq \limsup_{n\to\infty} \frac{f\lfloor \lambda n \rfloor}{f(n)} < \infty, \tag{43}$$

then for every $\Lambda > 1$, (43) holds uniformly in $\lambda \in [1, \Lambda]$:

$$0 < \liminf_{n\to\infty} \inf_{\lambda\in[1,\Lambda]} \frac{f(\lfloor \lambda n \rfloor)}{f(n)} \leq \limsup_{n\to\infty} \sup_{\lambda\in[1,\Lambda]} \frac{f(\lfloor \lambda n \rfloor)}{f(n)} < \infty.$$

My adaptation of the proof of this result appeals to knowledge of Lebesgue measure. This is the only place in the development of the theory where I have needed prerequisite knowledge at this level.

7. A Characterization Theorem

Galambos and Seneta [3] defined a sequence $(\theta(n))$ of positive numbers to be *regularly varying* if there exists a sequence $(\alpha(n))$ such that

$$\theta \sim K\alpha(n) \text{ and } n\left\{1 - \frac{\alpha(n-1)}{\alpha(n)}\right\} \to \rho \quad (n \to \infty)$$

where K is a positive constant and ρ is a finite constant. This is the discrete counterpart of the condition

$$R(x) \sim KR_1(x) \text{ where } \frac{xR_1'(x)}{R_1(x)} \to \rho \quad (x \to \infty)$$

for a regularly varying function R. Bojanic and Seneta [2], [1, p. 53] proved that this definition is equivalent to the usual one.

The following theorem establishes the expected counterpart for O-regularly varying sequences.

Theorem 6. *Let $f : \mathbb{N} \to \mathbb{R}^+$. Then $f \in OR_{seq}$ if and only if there exists $F : \mathbb{N} \to \mathbb{R}^+$ such that $f(n) \asymp F(n)$ and $n\left\{1 - \dfrac{F(n-1)}{F(n)}\right\}$ is bounded.*

Proof. First assume that $f \in OR_{seq}$, whence $f = \exp \circ (g + \tilde{h})$ with g, h bounded. Let $F(n) := \exp(\tilde{h}(n))$. Since $\exp(g(n)) \asymp 1$, we have $f(n) \asymp F(n)$. Check that

$$n\left\{1 - \frac{F(n-1)}{F(n)}\right\} = n\left\{1 - \frac{\exp(\tilde{h}(n-1))}{\exp(\tilde{h}(n))}\right\} = n\left\{1 - \exp\left(-\frac{h(n)}{n}\right)\right\} \tag{44}$$

where, since h is bounded, $|h(n)| \leq C$ $(n \in \mathbb{N})$ for some C, so that the above quantity lies between $n\{1 - \exp(\pm C/n)\}$, hence is bounded (a calculus exercise).

Next assume that $f \asymp F$ where $n\{1 - F(n-1)/F(n)\}$ is bounded. Let $h(n) := n\{1 - F(n-1)/F(n)\}$ $(n \in \mathbb{N})$; $F(0) := 1$. Check that

$$\frac{1}{F(n)} = \prod_{k=1}^{n} \frac{F(k-1)}{F(k)} = \prod_{k=1}^{n} \left\{1 - \frac{h(k)}{k}\right\} \tag{45}$$

and

$$\ln F(n) = \sum_{k=1}^{n} \frac{h(k)}{k} - \sum_{k=1}^{n} \left[\frac{h(k)}{k} + \ln\left\{1 - \frac{h(k)}{k}\right\}\right]. \tag{46}$$

Use calculus (series) to show that, if $|x| < 1$, then

$$|x + \ln(1-x)| \leq \frac{x^2}{1-|x|}. \tag{47}$$

Since h is bounded by hypothesis—say $|h(k)| \leq C$ for $k \in \mathbb{N}$ for some finite C—choose $M > C$, and show that

$$\left| \frac{h(k)}{k} + \ln\left\{1 - \frac{h(k)}{k}\right\}\right| \leq \frac{C^2/k^2}{1 - C/M} \quad (k \geq M). \tag{48}$$

Let

$$G(n) := \sum_{k=1}^{n} \left| \frac{h(k)}{k} + \ln\left\{1 - \frac{h(k)}{k}\right\}\right|.$$

Show that $G(n)$ is bounded (compare with $\sum 1/k^2$) and $\ln F(n) = G(n) + \tilde{h}(n)$. Let $g(n) := \ln\{f(n)/F(n)\} + G(n)$. Show that $g(n)$ is bounded (recall the hypothesis $f \asymp F$) and $\ln f(n) = g(n) + \tilde{h}(n)$, so that f is given by the representation formula of a O-regularly varying sequence. □.

8. A Discrete OR Tauberian Theorem

Tauberian theorems relate the asymptotic behavior of Laplace transforms of functions to the asymptotic behavior of the functions. The Laplace-Stieltjes transform of $U : [0, \infty) \to \mathbb{R}$ is given by

$$\widehat{U}(s) := \int_0^\infty e^{-sx} dU(x)$$

where, for our applications, U is nondecreasing and right-continuous. Karamata's Tauberian theorem is the following [1, p. 37].

Theorem 7. *Let $U : [0, \infty) \to \mathbb{R}$ be nondecreasing and right-continuous. Let $c \geq 0$, and let R be regularly varying with exponent $\rho \geq 0$. Then $U(x) \sim cR(x)/\Gamma(\rho+1)$ $(x \to \infty)$ if and only if $\widehat{U}(s) \sim cR(1/s)$ $(s \to 0+)$.*

The discrete version of this theorem involves power series rather than Stieltjes integrals, so the student need not have knowledge of such integrals. The version stated in Feller [4, p. 423], is the following.

Theorem 8. *Let $u : \mathbb{N}_0 := \mathbb{N} \cup \{0\} \to [0, \infty)$, let $U(n) := \sum_{k=1}^{n} u(k)$ for $n \in \mathbb{N}_0$, and assume that $\mathcal{U}(x) := \sum_{k=0}^{\infty} u(k)x^k$ converges for $x \in [0, 1)$. Then $\mathcal{U}(x) \sim R(x)$ as $x \to 1-$ where $R(x)$ is regularly varying at infinity with exponent $\rho \geq 0$ if and only if $U(n) \sim R(n)/\Gamma(\rho+1)$ as $n \to \infty$.*

The de Haan-Stadtmüller theorem ([7], [1, p. 118]) states a result like the Karamata Tauberian theorem for O-regularly varying functions.

Theorem 9. *Let $U : [0, \infty) \to \mathbb{R}^+$ be nondecreasing. The following are equivalent:*

(1) $U \in OR$;
(2) $\widehat{U}(1/\cdot) \in OR$;
(3) $\widehat{U}(1/t) \asymp U(t)$ $(t \to \infty)$.

Our discrete version of the de Haan-Stadtmüller/Karamata Tauberian theorem is the following.

Theorem 10. *Let $u : \mathbb{N}_0 \to [0, \infty)$, let $U(n) := \sum_{k=0}^{n} u(k)$ $(n \in \mathbb{N}_0)$, and assume that $\mathcal{U}(x) := \sum_{k=0}^{\infty} u(k)x^k$ converges for $x \in [0, 1)$. Let $(x(n))_{n=0}^{\infty}$ be a sequence in $(0, 1)$ such that $x(n) \to 1$ as $n \to \infty$ and $0 < c_1 \le x(n)^n \le c_2 < 1$ $(n \in \mathbb{N})$ for some constants c_1 and c_2. Then the following are equivalent.*

(1) $U \in OR_{seq}$;
(2) $\mathcal{U}(x(n)) \asymp U(n)$;
(3) $\mathcal{U}(x(\cdot)) \in OR_{seq}$.

For example, we may take $x(n) = 1 - 1/n$ or $x(n) = e^{-1/n}$.

The following lemma will come in handy.

Lemma 6. *If f is a nonnegative integrable function defined on $[1, \infty)$ that is nondecreasing on $[1, \xi]$ for some $\xi \ge 1$ and nonincreasing on $[\xi, \infty)$, then*

$$\sum_{k=m}^{\infty} f\left(\frac{k}{n}\right)\frac{1}{n} \le \int_{\frac{m}{n}}^{\infty} f(x)\,dx + \frac{f(\xi)}{n} \le \infty \quad (m, n \in \mathbb{N}, m \ge n).$$

Proof. Exercise. Consider a lower sum approximation for $\int_{m/n}^{\infty} f(x)\,dx$.

Proof of the Theorem. Preliminaries: Show that

$$(1 - x)\sum_{k=0}^{n} U(k)x^k = \sum_{k=0}^{n} u(k)x^k - U(n)x^{n+1} \tag{49}$$

and

$$\lim_{n\to\infty} U(n)x^{n+1} = 0 \quad (0 \le x < 1). \tag{50}$$

Therefore,

$$\mathcal{U}(x) = \sum_{k=0}^{\infty} u(k)x^k = (1 - x)\sum_{k=0}^{\infty} U(k)x^k \quad (0 \le x < 1). \tag{51}$$

Show that

$$\mathcal{U}(x) \ge x^n U(N) \text{ and } U(N) \le x^{-N}\mathcal{U}(x) \quad (N \in \mathbb{N}_0, 0 \le x < 1). \tag{52}$$

Let's start with an easy part, the proof that $\mathcal{U}(x(n)) \asymp U(n)$ implies $U \in OR_{seq}$. Since U is nondecreasing, by Corollary 1, it suffices to check that $U(2n)/U(n)$ is bounded. Use upper bounds on $U(N)$ (for suitable N) and on $x(n)^n$ to show that

$$\frac{U(2n)}{U(n)} \leq c_1^{-2} \frac{\mathcal{U}(x(n))}{U(n)}, \tag{53}$$

whence the desired conclusion follows.

Now let's prove that $U \in OR_{seq}$ implies $\mathcal{U}(x(n)) \asymp U(n)$. On the one hand, by (52), we have

$$\frac{\mathcal{U}(x(n))}{U(n)} \geq x(n)^n \geq c_1 \quad (n \in \mathbb{N}_0). \tag{54}$$

On the other hand, by (51), we have

$$\frac{\mathcal{U}(x(n))}{U(n)} = (1 - x(n)) \sum_{k=0}^{\infty} \frac{U(k)}{U(n)} x(n)^k, \tag{55}$$

where, since U is nondecreasing, $U(k)/U(n) \leq 1$ for $k \leq n$, and since $U \in OR_{seq}$, it satisfies the Potter-type bound

$$\frac{U(k)}{U(n)} \leq C \left(\frac{k}{n}\right)^{\alpha} \quad (k \geq n)$$

for some C and α. Now

$$\frac{\mathcal{U}(x(n))}{U(n)} \leq (1 - x(n)) \sum_{k=0}^{n} x(n)^k + (1 - x(n)) \sum_{k=n+1}^{\infty} C \left(\frac{k}{n}\right)^{\alpha} x(n)^k. \tag{56}$$

Furthermore, since $0 < c_1 \leq x(n)^n \leq c_2 < 1$ $(n \in \mathbb{N})$ and $\ln y \leq y - 1$ $(y > 0)$, we have

$$n\{1 - x(n)\} \leq -n \ln x(n) = -\ln x(n)^n \leq -\ln c_1 \in (0, \infty), \tag{57}$$

and $x(n)^k = \{x(n)^n\}^{k/n} \leq c_2^{k/n}$. Now show that

$$\frac{\mathcal{U}(x(n))}{U(n)} \leq 1 + \frac{(-\ln c_1)}{n} C \sum_{k=n+1}^{\infty} \left(\frac{k}{n}\right)^{\alpha} c_2^{k/n} \tag{58}$$

where, by Lemma 6, and letting $s = -\ln c_2$,

$$\sum_{k=n+1}^{\infty} \left(\frac{k}{n}\right)^{\alpha} c_2^{k/n} \leq \int_{(n+1)/n}^{\infty} x^{\alpha} e^{-sx}\, dx + \frac{c_3}{n} \tag{59}$$

where c_3 is an upper bound on $x^{\alpha} e^{-sx}$. Thence deduce that there is a finite upper bound c_4 such that

$$0 < c_1 \leq \frac{\mathcal{U}(x(n))}{U(n)} \leq c_4 < \infty. \tag{60}$$

Therefore, $U \in OR_{seq}$.

Now the proof that $U(\cdot) \in OR_{seq}$ implies $\mathcal{U}(x(\cdot)) \in OR_{seq}$ is easy. Since $U(\cdot) \in OR_{seq}$ implies $\mathcal{U}(x(n)) \asymp U(n)$, (already shown), we have

$$\frac{\mathcal{U}(x(\lfloor \lambda n \rfloor))}{\mathcal{U}(x(n))} \asymp \frac{U(\lfloor \lambda n \rfloor)}{U(n)} \asymp 1 \quad (\lambda \in \mathbb{R}^+),$$

so $\mathcal{U}(x(\cdot)) \in OR_{seq}$.

To finish, let's prove that $\mathcal{U}(x(\cdot)) \in OR_{seq}$ implies $U(\cdot) \in OR_{seq}$. Again, it suffices to show that $U(2n)/U(n)$ is bounded. Let $m, n \in \mathbb{N}$ with $m \geq n$. Use (51) and (52) and the nondecreasing nature of U to show that

$$\mathcal{U}(x) \leq (1-x) \sum_{k=0}^{m} U(m) x^k + (1-x) \sum_{k=m+1}^{\infty} x(k)^{-k} \mathcal{U}(x(k)) x^k. \tag{61}$$

Proceeding as in the proof of (58), show that

$$\mathcal{U}(x(n)) \leq U(m) + \frac{(-\ln c_1)}{n} c_1^{-1} C \mathcal{U}(x(n)) \sum_{k=m+1}^{\infty} \left(\frac{k}{n}\right)^{\alpha} c_2^{k/n}. \tag{62}$$

Letting $s = -\ln c_2$, show that

$$\sum_{k=m+1}^{\infty} \left(\frac{k}{n}\right)^{\alpha} c_2^{k/n} \leq \int_{m/n}^{\infty} x^{\alpha} e^{-sx}\, dx + \frac{\max\{x^{\alpha} e^{-sx}\}}{n}. \tag{63}$$

Hence, if n and m/n are sufficiently large—say $n \geq N$ and $m/n \geq A$—then

$$\mathcal{U}(x(n)) \leq U(m) + \frac{1}{2} \mathcal{U}(x(n)). \tag{64}$$

Therefore,

$$U(m) \geq \frac{1}{2} \mathcal{U}(x(n)) \quad (n \geq N, m \geq An).$$

On the other hand, $U(2m) \leq y^{-2m} \mathcal{U}(y)$ where $y = x(m)$ is allowed, and then $y^{-2m} = ((x(m)^m)^{-2} \leq c_1^{-2}$. Given $m \geq AN$, choose n such that $An \leq m < (A+1)n$. Then

$$\frac{U(2m)}{U(m)} \leq \frac{x(m)^{-2m} \mathcal{U}(x(m))}{\frac{1}{2} \mathcal{U}(x(n))} \leq 2 c_1^{-2} \frac{\mathcal{U}(x(m))}{\mathcal{U}(x(n))} \leq C' \left(\frac{m}{n}\right)^{\alpha'}$$

for suitable C' and α', by the Potter-type bounds for $\mathcal{U}(x(m))/\mathcal{U}(x(n))$. Here $m/n \leq A+1$, so $U(2m)/U(m)$ is bounded, and therefore, $U \in OR_{seq}$. □

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