The Probability a Shuffled Deck Has No Consecutive Cards of Equal Rank

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The following problem appeared in *M500*, the magazine of the British Open University mathematics student society, in 1991. (See [1].)

PROBLEM 124.1—CARD SHUFFLE

ANNE BATES

I have not been able to find a solution to the following problem. I have tried it out on a number of people, including an eminent professor of Applied Mathematics. All he could say was, "It is very difficult. You need a computer to work it out, and it would take the computer several hours." I still hope to find someone who can produce a *solution*, i.e., a method of finding the answer. The problem is:

If a standard pack of 52 cards is shuffled, what is the probability of finding two consecutive cards with the same value, such as two threes, two queens, etc., anywhere in the pack?

Binomial inversion

Our solution will use a formula of Bizley [2]. The key to this formula is the fact, easily checked, that the inverse of the matrix $[a_{i,j}]$ with $a_{i,j} = \binom{i}{j}$ is the matrix $[(-1)^{i-j}\binom{i}{j}]$. From this follows Bizley's binomial inversion formula: If, for all nonnegative values of $s_1, s_2, \ldots, s_n$,

$$U(s_1, s_2, \ldots, s_n) \;=\; \sum_{i_1=0}^{s_1}\sum_{i_2=0}^{s_2}\cdots\sum_{i_n=0}^{s_n}\binom{s_1}{i_1}\binom{s_2}{i_2}\cdots\binom{s_n}{i_n}F(i_1, i_2, \ldots, i_n),$$

then

$$F(s_1, s_2, \ldots, s_n) = \sum_{j_1=0}^{s_1}\sum_{j_2=0}^{s_2}\cdots\sum_{j_n=0}^{s_n}(-1)^{\sum s-\sum j}\binom{s_1}{j_1}\binom{s_2}{j_2}\cdots\binom{s_n}{j_n}U(j_1, j_2, \ldots, j_n).$$

Bizley's solution

We adapt the reasoning of Bizley [3] to the case at hand. To start with, we treat the number of cards of each rank, a_i, as a variable, and we regard the a_i cards within the i^{th} rank as indistinguishable. Let $f(a_1, a_2, \ldots, a_{13})$ denote the number of permutations of the $a_1 + a_2 + \cdots + a_{13}$ cards having no adjacent cards of equal rank. Now, the a_i cards of rank i can be divided into g_i groups having at least one card in each group in $\binom{a_i-1}{g_i-1}$ ways. Then, regarding each group as a unit, we can arrange the $\sum g_i$ groups so that no two groups of equal rank are adjacent in $f(g_1, g_2, \ldots, g_{13})$ ways. Then we may write the total number of ways of arranging the $a_1 + a_2 + \cdots + a_{13}$ cards without restriction in two (equal) ways:

$$\frac{(a_1 + a_2 + \cdots + a_{13})!}{a_1!a_2!\cdots a_{13}!} = \sum_{g_1=1}^{a_1}\sum_{g_2=1}^{a_2}\cdots\sum_{g_{13}=1}^{a_{13}}\binom{a_1-1}{g_1-1}\binom{a_2-1}{g_2-1}\cdots\binom{a_{13}-1}{g_{13}-1}f(g_1, g_2, \ldots, g_{13}).$$

By binomial inversion,

$$f(a_1, a_2, \ldots, a_{13}) =$$

$$\sum_{r_1=1}^{a_1} \sum_{r_2=1}^{a_2} \cdots \sum_{r_{13}=1}^{a_{13}} (-1)^{\sum a_j - \sum r_j} \binom{a_1 - 1}{r_1 - 1} \binom{a_2 - 1}{r_2 - 1} \cdots \binom{a_{13} - 1}{r_{13} - 1} \frac{(r_1 + r_2 + \cdots r_{13})!}{r_1! r_2! \cdots r_{13}!}.$$

Now, again regarding the cards within a rank as distinguishable, we find the number of arrangements with no two adjacent cards having the same rank by multiplying by $a_1! a_2! \cdots a_{13}!$, and then we obtain the corresponding probability by dividing by the number of permutations of $a_1 + a_2 + \cdots + a_{13}$ cards:

$$p(a_1, a_2, \ldots, a_{13}) = f(a_1, a_2, \ldots, a_{13}) \frac{a_1! a_2! \cdots a_{13}!}{(a_1 + a_2 + \cdots + a_{13})!}.$$

Standard deck

For a standard deck, $a_1 = a_2 = \ldots = a_{13} = 4$, and so the probability that no two adjacent cards have the same rank is

$$p(4, 4, \ldots, 4) = f(4, 4, \ldots, 4)(4!)^{13}/52!$$

$$= \sum_{r_1=1}^{4} \sum_{r_2=1}^{4} \cdots \sum_{r_{13}=1}^{4} (-1)^{52 - \sum r_j} \binom{3}{r_1 - 1} \binom{3}{r_2 - 1} \cdots \binom{3}{r_{13} - 1} \frac{(\sum r_j)!}{\prod r_j!} \frac{(4!)^{13}}{52!}$$

Simplification

There are 4^{13} (approximately 65 million) terms in this sum, and so it might seem very difficult to evaluate this result numerically. But there are actually fewer than one thousand *different* terms, and so it is not difficult at all. Given $(r_1, r_2, \ldots, r_{13})$, let $t_i = \#\{j : r_j = i\}$ $(i = 1, 2, 3, 4)$. The value of a term turns out to depend only on (t_1, t_2, t_3, t_4).

Notice that $\sum_{j=1}^{13} r_j = 1 \cdot t_1 + 2 \cdot t_2 + 3 \cdot t_3 + 4 \cdot t_4$. So

$$\begin{aligned}(-1)^{52-\sum r_j} &= (-1)^{-\sum r_j} \\ &= (-1)^{\sum r_j} \\ &= (-1)^{1\cdot t_1+2\cdot t_2+3\cdot t_3+4\cdot t_4} \\ &= (-1)^{t_1+t_3},\end{aligned}$$

$$\begin{aligned}\binom{3}{r_1-1}\binom{3}{r_2-1}\cdots\binom{3}{r_{13}-1} &= \binom{3}{1-1}^{t_1}\binom{3}{2-1}^{t_2}\binom{3}{3-1}^{t_3}\binom{3}{4-1}^{t_4} \\ &= 3^{t_2+t_3},\end{aligned}$$

and

$$\frac{(\sum r_j)!}{\prod r_j!} = \frac{(t_1 + 2 \cdot t_2 + 3 \cdot t_3 + 4 \cdot t_4)!}{1!^{t_1} 2!^{t_2} 3!^{t_3} 4!^{t_4}}.$$

We'll decompose the sum according to the (t_1, t_2, t_3, t_4) values. Since $t_1 + t_2 + t_3 + t_4 = 13$, we may sum over, say, (t_2, t_3, t_4). Then

$$\begin{aligned}(-1)^{t_1+t_3} &= (-1)^{13-t_2-t_3-t_4+t_3} \\ &= (-1)^{13-t_2-t_4} \\ &= (-1)^{13}(-1)^{-t_2-t_4} \\ &= (-1)^{1+t_2+t_4}.\end{aligned}$$

Also, $t_1 + 2 \cdot t_2 + 3 \cdot t_3 + 4 \cdot t_4 = 13 - t_2 - t_3 - t_4 + 2 \cdot t_2 + 3 \cdot t_3 + 4 \cdot t_4 = 13 + t_2 + 2t_3 + 3t_4$, each $t_i \in \{0, 1, \ldots, 13\}$, and $t_2 + t_3 + t_4 \leq 13$. Finally, there are $13!/t_1!t_2!t_3!t_4!$ 13-tuples $(r_1, \ldots, r_{13})$ corresponding to a particular 4-tuple value (t_1, t_2, t_3, t_4), and so

$$p(4, 4, \ldots, 4) = \sum_{t_2=0}^{13} \sum_{t_3=0}^{13-t_2} \sum_{t_4=0}^{13-t_2-t_3} \frac{13!(-1)^{1+t_2+t_4} 3^{t_2+t_3}}{(13 - t_2 - t_3 - t_4)! t_2! t_3! t_4!} \frac{(13 + t_2 + 2t_3 + 3t_4)!}{2!^{t_2} 3!^{t_3} 4!^{t_4}} \frac{(4!)^{13}}{52!}.$$

Solution of problem 124.1

By means of a *Mathematica* program running on a NeXT workstation, the value specified by the above formula was found to be about 0.0454763 in five seconds of computing time. Thus the probability of finding two consecutive cards with the same value somewhere in the pack is about 95.5%.

Acknowledgments

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References

[1] Anne Bates, Problem 124.1–card shuffle, *M500*, 124 (1991), 21.

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[3] M.T.L. Bizley, A problem in permutations, *The American Mathematical Monthly*, 70 (1963), 722–730.

[4] John Holte and Mark Holte, The probability of n ace-king adjacencies in a shuffled deck, *Abstracts of the AMS*, Vol. 14, No. 5 (August 1993), p. 60, 883-60-234.