The Probability a Queen Sits beside a King: Juxtapositions and Runs in a Random Permutation

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Introduction

Suppose that you thoroughly shuffle a deck of ordinary playing cards and then deal them out in one long row. What is the probability that somewhere in the row you will find a queen next to a king? Or a club next to a diamond? Or a face card next to an ace? More generally, in a random permutation of n objects of various kinds, what is the probability of finding j instances where an object of a given kind is adjacent to one of another given kind?

At first glance you may conclude that an explicit solution would be quite difficult. Indeed, an earlier published solution [Singmaster 1991] for the special case of *no* queen-king juxtapositions relied entirely on a recurrence-relation calculation. But in fact explicit solutions to such problems turn out to be elementary and accessible to undergraduate students who have studied combinatorics or discrete probability. Furthermore, the juxtaposition problem turns out to be closely related to the problem of determining the distribution of runs, which is studied in many undergraduate mathematical statistics courses and whose solution has been known at least since 1886 (see [Whitworth 1886, Exercises 193, 194], [Barton and David 1957, 168]).

In fact, the solution of the juxtaposition problem to be given here provides a natural follow-up to the solution of the runs problem.

In the special case of ace-king juxtapositions, an explicit solution for j juxtapositions was given previously in [Holte and Holte 1993]. An explicit solution for the general problem of no adjacencies between any objects of k different kinds was given in [Suman 1993]. In this module, we extend the techniques of these papers to find the probability of exactly j juxtapositions occurring between objects of two different kinds in a random permutation of any number of objects of two or more kinds. In addition, we find the mean and variance of the number of juxtapositions. Doing this provides a nice illustration of techniques for dealing with moments of sums of dependent random variables.

The probability a queen is next to a king

Let us begin by finding the probability that there are exactly j places where a queen is adjacent to a king *in the order specified* in a shuffled deck. For now, assume that all four queens (Q's) are identical, all four kings (K's) are identical, and all 44 cards that are neither (N's) are identical. Clearly, if we can find the number of all permissible permutations under this assumption, then we can multiply this answer by 4!4!44! to adjust for the fact that the cards are all distinct. Now a permutation of the deck might look like this:

$$\text{NNNNNN}Q\text{NNNNN}QK\text{NNNN}Q\text{NNNNNNNN}K\text{NNNNNNNNN}K\text{NNNN}Q\text{NNN}K\text{NNNNN}.$$

Suppressing the N's, we see

$$Q\ QK\ Q\ K\ K\ Q\ K.$$

Without the N's there are three potential QK juxtapositions, but with the N's inserted as above, there was just one such actual juxtaposition.

Let's count the number of arrangements of the four Q's and four K's that by themselves have m (potential) QK juxtapositions. We first consider the Q's alone and mark the m places where K's will follow a Q; in our example with $m = 3$, we'd write, e.g.,

$$\text{Q Q} \mid \text{Q} \mid \text{Q} \mid .$$

There are $\binom{4}{m}$ ways to choose these places. Second we consider the K's alone and mark the m places where a K is to be preceded by Q's; e.g.,

$$| \text{ K } | \text{ K K } | \text{ K } .$$

There are $\binom{4}{m}$ ways to choose these places. With these choices, the string of Q's and K's is determined: it would be $QQKQKKQK$ in our example. There are $\binom{4}{m}^2$ possible arrangements.

Next, let's count the number of ways we may insert the N's so as to achieve j actual juxtapositions. Think of the nine spaces before, between, and after the four Q's and four K's as urns. To achieve j actual juxtapositions, choose j of the m "QK" urns to be empty; this may be done in $\binom{m}{j}$ ways. To make sure that there are no more than j juxtapositions, place one N in each of the other $m-j$ "QK" urns. Then distribute the remaining $44-(m-j)$ N's into the $9-j$ urns that do not have to be empty. We need to know here and later that *the number of ways to place* r *identical balls into* u *urns without restriction is* $\binom{r+u-1}{u-1}$. (This formula may be established by a "stars-and-bars" argument: Write $*$ for each ball, and divide them into u groups by $u-1$ $|$'s. For example, $***|**||*$ represents 3 balls in the first urn, 2 in the second, none in the third, and 1 in the fourth. The number of ways to choose $u-1$ places from $r+u-1$ for the $|$'s is $\binom{r+u-1}{u-1}$.) For us, the number of ways we may distribute the N's is

$$\binom{(44-(m-j))+(9-j)-1}{(9-j)-1} = \binom{52-m}{8-j}.$$

Thus there are $\binom{4}{m}^2\binom{m}{j}\binom{52-m}{8-j}$ arrangements of Q's, K's, and N's having m potential QK juxtapositions and j actual QK juxtapositions. Multiplying by $4!4!44!$, summing over all possible values of m ($j \leq m \leq 4$), and dividing by the number of permutations of 52 cards, we get the probability q_j of j QK juxtapositions:

$$q_j = \sum_{m=j}^{4} \binom{4}{m}^2 \binom{m}{j} \binom{52-m}{8-j} \frac{4!4!44!}{52!}.$$

Approximate values of q_j are given in the following table.

j	0	1	2	3	4
q_j	0.7187	0.2556	0.0250	0.0007	4×10^{-6}

The probability of j queen-king juxtapositions in either order (QK or KQ) will be computed in the next section.

Exercises

1. List the $\binom{4}{3}^2$ possible sequences of 4 Q's and 4 K's having exactly 3 "QK" juxtapositions.

2. Suppose that the deck is reduced to 2 Q's, 2 K's, and one ace. Adapt the analysis in the text to complete the following table. Find the probability q_j of j QK juxtapositions for each j.

m	Q's to follow	K's to precede	K-Q sequence	# ways to insert A to get		
				$j = 0$	$j = 1$	$j = 2$
$m = 0$	QQ	KK	$KKQQ$			
$m = 1$	$Q\mid Q$	$\mid KK$		1	4	0
$m = 1$	$Q\mid Q$	$K\mid K$	$KQKQ$			
$m = 1$	$QQ\mid$	$\mid KK$		1		
$m = 1$	$QQ\mid$	$K\mid K$	$KQQK$			
$m = 2$	$Q\mid Q\mid$	$\mid K\mid K$	$QKQK$			

Juxtapositions and runs

Now consider a set of n distinguishable objects of various kinds: a of one kind (A's), b of a second kind (B's), and $c = n - a - b$ of other kinds (collectively C's). What is the probability that a random permutation of the n objects contains exactly j instances where an object of the first kind is adjacent (on either side) to an object of the second kind?

Temporarily regard the A's (respectively, B's) as indistinguishable. Let A^+ denote a generic string of one or more consecutive A's; define B^+ similarly. When our a A's and b B's are laid out in a line, the A^+'s and B^+'s alternate, so the numbers of A^+'s and B^+'s must be either the same or differ by one; let i be the common or smaller number, as the case may be. Then the following table summarizes all the possibilities.

Pattern	$\#A^+$'s	$\#B^+$'s	$\#AB$'s	$\#BA$'s	#juxtapositions
$A^+B^+A^+\cdots A^+B^+$	i	i	i	$i-1$	$2i-1$
$A^+B^+A^+\cdots B^+A^+$	$i+1$	i	i	i	$2i$
$B^+A^+B^+\cdots A^+B^+$	i	$i+1$	i	i	$2i$
$B^+A^+B^+\cdots B^+A^+$	i	i	$i-1$	i	$2i-1$

Each A^+ and B^+ represents a "run" of A's or B's, respectively. A string has m AB and BA juxtapositions if and only if it has $m+1$ runs of A's and B's. Let $R_k = R_k(a,b)$ denote the number of arrangements of a A's and b B's yielding k runs of A's and B's. Then the number of arrangements of a A's and b B's yielding exactly m AB or BA juxtapositions is $R_{m+1}(a,b)$. The formula (1) given below is well known in the theory of runs—see, e.g., [David and Barton 1962, ch. 6], [Freund 1992, 594–595], [Hogg and Craig 1995, 517–519], [Hogg and Tanis 1993, sect. 10.6], or [Ross 1994, 47–49, 57–58]—and the reader familiar with the result may jump ahead to that point, but for completeness it will be rederived next. Values of the probabilities of k runs—$r(k;a,b) = R_k(a,b)/\binom{a+b}{a}$—may be found in statistical tables, e.g., [Beyer 1968, Table X.6 (414–424)], [Swed and Eisenhart 1943].

Now let's count the ways to get the patterns in the table. In the first pattern, $A^+B^+A^+\cdots A^+B^+$, after putting a B^+ after the last A, we must insert the other $i-1$ B^+'s after $i-1$ A's (not A^+'s), which may be chosen in $\binom{a-1}{i-1}$ ways. This determines the A's that are to be followed by B^+'s, but it does not determine the B's that are to be followed by A^+'s. Aside from the last B, there are $b-1$ B's from which we must choose $i-1$; this may be done in $\binom{b-1}{i-1}$ ways. Altogether, then, we have $\binom{a-1}{i-1}\binom{b-1}{i-1}$ ways to achieve the $A^+B^+A^+\cdots A^+B^+$ pattern.

For the second pattern, $A^+B^+A^+\cdots B^+A^+$, we must use at least one A at the end, and then insert i B^+'s after i A's, which may be chosen in $\binom{a-1}{i}$ ways. Then, aside from the last B, which is necessarily followed by A^+, we must choose $i-1$ B's to be followed immediately by A^+'s, which can be done in $\binom{b-1}{i-1}$ ways. Altogether, then, we have $\binom{a-1}{i}\binom{b-1}{i-1}$ ways to achieve the $A^+B^+A^+\cdots B^+A^+$ pattern.

The remaining patterns may be counted by switching A and B (and, correspondingly, a and b) and applying the formulas just derived. The result is that the number of $B^+A^+B^+\cdots A^+B^+$ patterns is $\binom{b-1}{i}\binom{a-1}{i-1}$, and the number of $B^+A^+B^+\cdots B^+A^+$ patterns is $\binom{b-1}{i-1}\binom{a-1}{i-1}$.

If $m = 2i-1$ (first and fourth patterns), then the number of patterns having m juxtapositions is given by

$$R_{m+1} = R_{2i} = 2\binom{a-1}{i-1}\binom{b-1}{i-1}.$$

If $m = 2i$ (second and third patterns), then the number of patterns having m juxtapositions is given by

$$R_{m+1} = R_{2i+1} = \binom{a-1}{i}\binom{b-1}{i-1} + \binom{a-1}{i-1}\binom{b-1}{i}.$$

We may combine the odd- and even-case formulas by using the greatest-integer function, denoted here by $\lfloor\cdot\rfloor$. If $m = 2i-1$, then $\lfloor m/2\rfloor = i-1$ and $\lfloor (m-1)/2\rfloor = i-1$, and if $m = 2i$, then $\lfloor m/2\rfloor = i$ and $\lfloor (m-1)/2\rfloor = i-1$. Thus, the number of arrangements of a A's and b B's having m AB or BA juxtapositions is given by

$$R_{m+1}(a,b) = \binom{a-1}{\lfloor m/2\rfloor}\binom{b-1}{\lfloor (m-1)/2\rfloor} + \binom{a-1}{\lfloor (m-1)/2\rfloor}\binom{b-1}{\lfloor m/2\rfloor}. \quad (1)$$

Now, given an arrangement of a A's and b B's yielding m potential AB or BA juxtapositions, we insert the remaining c objects (C's) so as to achieve exactly j juxtapositions. There are $a+b+1$ "urns" available for these other objects, and we must keep empty j of the m "AB" or "BA" urns. There are $\binom{m}{j}$ ways to choose these urns. To make sure that there are no more than j juxtapositions, we place one C in each of the other $m-j$ "AB" and "BA" urns. Then we distribute the remaining $c-(m-j)$ C's into the $a+b+1-j$ urns that do not have to be empty; the number of ways this can be done is

$$\binom{(c-(m-j)) + (a+b+1-j) - 1}{(a+b+1-j)-1} = \binom{n-m}{a+b-j}.$$

Because there are $\binom{n}{a+b}$ equally likely ways to choose positions for the A's, B's, and C'S, the conditional probability of achieving j juxtapositions is

$$\frac{\binom{m}{j}\binom{n-m}{a+b-j}}{\binom{n}{a+b}} = h(j; a+b, m, n-m),$$

i.e., a hypergeometric probability. (Note: $h(j; a+b, m, n-m) = h(m-j; c, m, n-m)$.

Finally, assuming that all $n!$ permutations are equally likely and taking into account the possible permutations of the A's, B's, and C's, we find that the probability p_j of getting exactly j instances where an A-object is next to a B-object (in either order) is

$$p_j = \sum_m R_{m+1}(a,b) \binom{m}{j} \binom{n-m}{a+b-j} \frac{a!b!c!}{n!} \tag{2}$$

where $a+b+c=n$ and the sum is taken over all possible values of m ($\max\{j,1\} \le m \le 2\min\{a,b\}$) and $R_{m+1}(a,b)$ is given by equation (1). Alternatively, p_j may be expressed in terms of run probabilities and hypergeometric probabilities:

$$p_j = \sum_{m=\max\{j,1\}}^{2\min\{a,b\}} r(m+1;a,b)h(j;a+b,m,n-m).$$

Example 1. The probability that, in an ordinary shuffled deck, we have j queen-king or king-queen juxtapositions is given by equation (2) with $n=52$ and $a=b=4$. Approximate values are given in the following table. Note that the probability that some queen is next to some king (on either side) is $1-p_0 \approx 0.486$. This value was found via recurrence relations in [Singmaster 1991].

j	0	1	2	3	4	5	6	7
p_j	.514	.372	.100	.013	.001	3×10^{-5}	4×10^{-7}	2×10^{-9}

Example 2. The probability that in an ordinary shuffled deck we have j instances where a club is next to a diamond is given by equation (2) with $n=52$ and $a=b=13$. Approximate values that are at least 0.01 are given in the following table.

j	2	3	4	5	6	7	8	9	10	11	12
p_j	.01	.04	.10	.16	.19	.19	.14	.09	.04	.02	.01

Example 3. The probability that there are j instances where an ace is next to a face card is given by equation (2) with $n=52$, $a=4$, and $b=12$. Approximate values are given in the following table.

j	0	1	2	3	4	5	6	7	8
p_j	.114	.296	.320	.189	.066	.014	.002	.000	.000

Exercises

3. Consider the $5!/(2!2!1!)$ five-letter "words" that can be formed using 2 A's, 2 B's, and 1 C. For each i (the smaller of the number of A- and B-runs) and for each pattern in the table of possibilities, list the five-letter words that give each number j of AB and BA juxtapositions. Confirm that $p_0 = 1/15$, $p_1 = p_2 = 2/5$, and $p_3 = 2/15$.

4. Three contagious people, four susceptible people, and five immune people line up randomly. What is the probability that a contagious person stands next to a susceptible person in the queue?

5. Six bar magnets and 6 bars of metal are inserted at random and pushed together in a horizontal plastic pipe. Suppose that the orientation of 3 of the bar magnets is N-S, and for the other 3 it is S-N. When like magnetic poles are adjacent, the magnets will repel. For $j = 0, \ldots, 5$, what is the probability that there are j gaps?

Ordered juxtapositions

The preceding analysis can be adapted to the case where we count juxtapositions only if they occur in a particular order, A to the left of B, say. If we replace i by $i+1$ in the $B^+A^+B^+ \cdots B^+A^+$ row of the table of patterns, so that the number of AB juxtapositions stipulated there is i (as it is in all the other rows), then we find that the number of ways to arrange a A's and b B's so as to get $m = i$ AB juxtapositions is

$$\binom{a-1}{i-1}\binom{b-1}{i-1}+\binom{a-1}{i}\binom{b-1}{i-1}+\binom{b-1}{i}\binom{a-1}{i-1}+\binom{b-1}{i}\binom{a-1}{i} = \binom{a}{i}\binom{b}{i}.$$

The simpler answer points to an easier way to get this, which is left as an exercise for the reader. Now the probability q_j of getting j AB's is

$$q_j = \sum_{m=j}^{\min\{a,b\}} \binom{a}{m}\binom{b}{m}\binom{m}{j}\binom{n-m}{a+b-j}\frac{a!b!c!}{n!}.$$

Example 4. If the letters of the word "STATISTICS" are written on ten cards that are then permuted randomly, then the probability q_j of getting j ST's is given as follows.

j	0	1	2	3
q_j	$7/24 = .291\overline{6}$	$.21/40 = 5250$	$7/40 = .1750$	$1/120 = .008\overline{3}$

Exercises

6. Give a combinatorial argument to explain why the the number of ways to arrange a A's and b B's so as to get exactly i "AB" juxtapositions is $\binom{a}{i}\binom{b}{i}$.

7. Suppose that 6 print jobs in a computer system are queued at random. Two are long, two are of medium length, and two are short. What is the probability that no long job immediately precedes a short job?

8. Of the 30 students enrolled in a statistics class, 10 are female math majors (F's), 12 are male math majors (M's), and 8 are non-math majors (N's). As the students turned in their midterm exams, the professor noticed that quite often a male math major turned in his exam right after a female math major turned in hers. Thus the professor was interested in testing

H_0: The sequence of 10 F's, 12 M's, and 8 N's is random versus

H_1: There are more FM adjacencies than would be expected under H_0.

The order in which they turned in their midterm exams was

$$FMNNFFMFMNMNMFMMFFMNNMFMNMFMNF.$$

There are 7 FM's. Find $P($ 7 or more FM adjacencies $|H_0$ is true). Should one reject H_0 at the 5% level of significance?

Mean and variance

In order to get a better sense of the central tendency and spread of the possible numbers of juxtapositions, let us calculate the mean and variance of the random variable T which is defined as the total number of times that, in a random permutation of a set of n distinguishable objects, an object of a specified subset of a objects (A's) is adjacent (on either side) to an object from a specified disjoint subset of b objects (B's). We claim that the mean, or expected value of T, is

$$\mu = E(T) = 2ab/n$$

and the variance of T is

$$\sigma^2 = \frac{2ab}{n}\left(1 - \frac{a+b}{n-1} + \frac{2ab}{n(n-1)}\right) = \mu\left(1 - \frac{a+b-\mu}{n-1}\right) = \frac{\mu(\mu+c-1)}{n-1},$$

where $c = n - (a+b)$.

Thus, for example, when T is the number of queen-king or king-queen juxtapositions, we have $n = 52$ and $a = b = 4$, so $\mu = 2 \cdot 4 \cdot 4/52 = 8/13$, $\sigma^2 = (8/13)(1-(4+4-8/13)/51) = 1512/2873$, and the standard deviation is about 0.725. If we shuffled d ordinary decks together, then the mean number of such juxtapositions would be $2 \cdot 4d \cdot 4d/(52d) = (8/13)d$ and the standard deviation would be $((8d/13)(1-(8d-8d/13)/(52d-1)))^{1/2} \approx 0.73\sqrt{d}$.

In the case that $a+b = n$, the mean and variance formulas are well known from the theory of runs: see, e.g., [David and Barton 1962, 88], [Freund 1992, 596], [Hogg and Tanis 1993, 634], or [Ross 1994, 313–314] (mean only). Still, we shall give an independent derivation of these results—one that works for $a + b \leq n$.

Rather than undertake the daunting task of calculating the mean and variance directly from equation (2), let us use the standard technique of writing T as a sum of indicator random variables. Introduce the random variable X_i that is defined (for $1 \leq i \leq n-1$) to be 1 if AB or BA occurs in positions i & $i+1$ and that is defined to be 0 otherwise. Then $T = \sum X_i$.

We observe that $E(X_i) = 1 \cdot P(X_i = 1)$, the probability of a juxtaposition in positions i & $i+1$, which is $(a/n)(b/(n-1)) + (b/n)(a/(n-1))$. By the linearity property of expectation, or mean value, we obtain

$$E(T) \quad = \quad E\left(\sum_{i=1}^{n-1} X_i\right)$$

$$\begin{aligned}
&= \sum_{i=1}^{n-1} E(X_i) \\
&= (n-1)\left(\frac{a}{n}\cdot\frac{b}{n-1}+\frac{b}{n}\cdot\frac{a}{n-1}\right) \\
&= \frac{2ab}{n}.
\end{aligned}$$

The calculation of the variance is quite a bit more difficult, and serves as a good illustration of the sort of calculation one must go through when finding the variance of a sum of correlated random variables, when the answer is not simply the sum of the variances—the sort of thing one does, for example, to get the variance of a hypergeometric distribution. We begin with

$$\sigma^2 = E(T^2) - (E(T))^2$$

and

$$E(T^2) = E\left(\sum X_i \sum X_j\right) = \sum\sum E(X_i X_j).$$

Now $E(X_iX_j) = 1 \cdot P(X_i = 1 \& X_j = 1)$, the probability of juxtapositions in positions i & $i+1$ and j & $j+1$. The possibilities for the $(n-1)^2$ terms are given in the following table.

Indices	#terms	Patterns	$E(X_iX_j)$
$j = i$	$n-1$	AB BA	$\frac{a}{n}\cdot\frac{b}{n-1}+$ $\frac{b}{n}\cdot\frac{a}{n-1}$
$j = i+1$ or $j = i-1$	$2(n-2)$	ABA BAB	$\frac{a}{n}\cdot\frac{b}{n-1}\cdot\frac{a-1}{n-2}+$ $\frac{b}{n}\cdot\frac{a}{n-1}\cdot\frac{b-1}{n-2}$
$j \geq i+2$ or $j \leq i-2$	$(n-2)(n-3)$	AB,AB AB,BA BA,AB BA,BA	$4\cdot\frac{a}{n}\cdot\frac{b}{n-1}\cdot\frac{a-1}{n-2}\cdot\frac{b-1}{n-3}$
$1 \leq i, j \leq n-1$	$(n-1)^2$		

Adding every $E(X_iX_j)$ value multiplied by its number of terms and simplifying, we get

$$E(T^2) = \frac{2ab}{n}\left(1+\frac{2ab-a-b}{n-1}\right).$$

Then $\sigma^2 = E(T^2) - (E(T))^2$ reduces to the result claimed above, completing the calculation.

Exercises

9. What is the expected number of club-diamond and diamond-club juxtapositions in a randomly shuffled deck? What is the variance?

10. In a randomly shuffled deck, how many runs of face cards will there be on average? What is the standard deviation?

11. Suppose that when the scores of college students on a test are arranged in order (assume no ties) and the class of each student is noted ($1 =$ freshman, $2 =$ sophomore, etc.), the following sequence of classes is observed:

$$3\ 2\ 3\ 111\ 4\ 1\ 44\ 2\ 4\ 2\ 44\ 22\ 11\ 3$$

Is the number of runs in this sequence greater than would be expected by chance?

12. Let U denote the number of "AB" juxtapositions (in that order) in a random permutation of a A's, b B's, and c C's. Derive formulas for the mean and variance of U.

13. The number of runs is used as a nonparametric statistic in tests of randomness. The number of juxtapositions may be used instead. For example, suppose that a row of 14 plants shows the following pattern of 7 healthy (H) and 7 diseased (D) plants:

$$HHHDDDDHHHHDDD.$$

Is the arrangement random?

(a) Calculate the mean and standard deviation of the number of HD and DH juxtapositions, and interpret the results.

(b) Calculate the probability of getting 3 or fewer juxtapositions if the sequence is random.

(c) Calculate the probability of getting 10 or more juxtapositions if the sequence is random. Notice that the maximum possible number of juxtapositions is 13, and $13 - 3 = 10$.

(d) Verify that the probability of getting a number of juxtapositions as extreme as the number in the sample is about 5%.

14. Five men, five women, and ten children get into a line.

(a) If they line up at random, what is the mean and standard deviation of the number of woman-child/child-woman juxtapositions?

(b) A statistician suspects that children will tend to get next to women in line. What is the probability of getting at least as many cases of this as in the following sample?

$$WMCCWCWCMCCWCMCWCMCM$$

Does the evidence confirm the statistician's suspicion?

Related problems

The juxtaposition problems we have considered represent just one kind of problem connected with random permutations. Besides the run probabilities considered here, there are many interesting problems in the distribution theory of runs, including those involving runs of more than two kinds of objects; comprehensive accounts are given in the book [David and Barton 1962] and the classic article [Mood 1940]. The former source also discusses matching and derangement problems for (two) random permutations, and it gives extensive calculations of moments via indicator random variables. Takács [1981] also discusses the classical problem of ménages—finding the probability that some husband and wife will be seated together if n couples are randomly seated around a circular table with men and women alternating. Barbour *et al.* [1992, ch. 4] address various problems connected with random permutations, including matching and ménage problems; they calculate moments via indicator random variables, and they provide Poisson approximations to various probability distributions. But none of these sources give the juxtaposition probabilites presented in this module.

Random Circular Permutations

The concluding exercises develop the theory and applications of random circular permutations. Suppose that n letters—a A's, b B's, and c C's—are randomly permuted in a line, and then the line is bent so that the ends meet to form a ring.

Exercises

15. Derive formulas for the mean and variance of the number U of "AB" juxtapositions.

16. Derive formulas for the mean and variance of the number T of "AB" and "BA" juxtapositions.

17. Derive a formula for the probability of exactly j "AB" juxtapositions in a random circular permutation in terms of such probabilities for linear permutations.

18. Derive a formula for the probability of exactly j "AB" and "BA" juxtapositions when a A's, b B's, and c C's are randomly permuted and bent into a ring.

19. Three Arabs, four Israelis, and six Swiss are seated randomly at a circular table. What are the mean and variance of the number of Arab-Israeli juxtapositions? What is the probability that no Arab and no Israeli are side by side?

20. Repeat Examples 1–4 for circular permutations.

Solutions

1. $KQKQKQKQ$ $KQKQKQQK$ $KQKQQKQK$ $KQQKQKQK$
   $QKKQKQKQ$ $QKKQKQQK$ $QKKQQKQK$ $QQKKQKQK$
   $QKQKKQKQ$ $QKQKKQQK$ $QKQQKKQK$ $QQKQKKQK$
   $QKQKQKKQ$ $QKQKQQKK$ $QKQQKQKK$ $QQKQKQKK$

2.

m	Q's to follow	K's to precede	K-Q sequence	# ways to insert A to get		
				$j=0$	$j=1$	$j=2$
$m=0$	QQ	KK	$KKQQ$	5	0	0
$m=1$	$Q\|Q$	$\|KK$	$QKKQ$	1	4	0
$m=1$	$Q\|Q$	$K\|K$	$KQKQ$	1	4	0
$m=1$	$QQ\|$	$\|KK$	$QQKK$	1	4	0
$m=1$	$QQ\|$	$K\|K$	$KQQK$	1	4	0
$m=2$	$Q\|Q\|$	$\|K\|K$	$QKQK$	0	2	3

Therefore $q_0 = 9/30 = 3/10$, $q_1 = 18/30 = 3/5$, and $q_2 = 3/30 = 1/10$.

3.

$i=1$	AABB	$j=0$: AACBB. $j=1$: AABBC, AABCB, ACABB, CAABB.
$i=1$	ABBA	$j=1$: ABBCA, ACBBA. $j=2$: ABBAC, ABCBA, CABBA.
$i=1$	BAAB	$j=2$: BAACB, BCAAB. $j=2$: BAABC, BACAB, CBAAB.
$i=1$	BBAA	$j=0$: BBCAA. $j=1$: BBAAC, BBACA, BCBAA, CBBAA.
$i=2$	ABAB	$j=2$: ABACB, ABCAB, ACBAB. $j=3$: ABABC, CABAC.
$i=2$	BABA	$j=2$: BABCA, BACBA, BCABA. $j=3$: BABAC, CBABA.

4. Let type A be contagious; B, susceptible; and C, immune; then $a = 3, b = 4, c = 5$, and $n = 12$. We want to calculate $1 - p_0$. By the formula given,

$$p_0 = \left[2 \cdot \binom{11}{7} + 5 \cdot \binom{10}{7} + 12 \cdot \binom{9}{7} + 9 \cdot \binom{8}{7} + 6 \cdot \binom{7}{7}\right] \frac{3!4!5!}{12!} = \frac{59}{924},$$

so $1 - p_0 \approx 94\%$.

5.

j	0	1	2	3	4	5
p_j	$\frac{61}{440} \approx .139$	$\frac{101}{264} \approx .383$	$\frac{223}{660} \approx .338$	$\frac{27}{220} \approx .123$	$\frac{23}{1320} \approx .017$	$\frac{1}{1320} \approx .001$

6. There are $\binom{a}{i}$ ways to choose the i A's that will be followed by one or more B's, and there are $\binom{b}{i}$ ways to choose the i B's that will be preceded by one or more A's. These two choices uniquely determine an interleaving of the A's and B's yielding i "AB" juxtapositions, so altogether there are $\binom{a}{i}\binom{b}{i}$ ways to achieve i juxtapositions.

7. Here $a = 2$ (long), $b = 2$ (short), and $c = 2$, so the answer is $q_0 = 2/5$.

8. $P($ 7 or more FM adjacencies $|H_0$ is true$) = \sum_{j=7}^{10} q_j \approx 0.024 < 0.05$, so we reject H_0 at the 5% level.

9. Here $a = 13, b = 13, c = 26$, and $n = 52$, so the expected number is $\mu = 2ab/n = 2 \cdot 13 \cdot 13/52 = 13/2 = 6.50$ and the variance is $\mu(\mu + c - 1)/(n - 1) = (13/2)(13/2 + 26 - 1)/51 = 273/68 \approx 4.01$.

10. The number of runs is $T + 1$ where T is the number of face-nonface and nonface-face juxtapositions. Here $a = 12, b = 40, c = 0$, and $n = 52$, so $E(\#runs) = E(T + 1) = E(T) + 1 = \mu + 1$ where $\mu = 2 \cdot 12 \cdot 40/52 = 240/13$, so $\mu + 1 = 253/13 \approx 19.46$ and $\sigma^2_{T+1} = \sigma^2_T = \mu(\mu + c - 1)/(n - 1) = 18160/2873$ and $\sigma_T \approx 2.51$.

11. There are 6 freshman, 5 sophomores, 3 juniors, and 6 seniors. Let T_{ij} denote the number of ij and ji juxtapositions, and let R denote the total number of runs of identical digits. Then $R = 1 + T_{12} + T_{13} + T_{14} + T_{23} + T_{24} + T_{34}$. Each $E(T_{ij}) = 2ab/n$ for appropriate a and b, so $E(R) = 1+2[(6)(5)+(6)(3)+(6)(6)+(5)(3)+(5)(6)+(3)(6)]/20 = 15.7$. The observed number of runs is 14—just a little below what is expected. (The theory presented here is not adequate to compute $P(R \leq 14)$—or even the variance of R, because of the covariances involved.)

12. Here let $X_i = 1$ if the i^{th} value is an "A" and the $i + 1^{th}$ value is a "B," and let $X_i = 0$ otherwise $(1 \leq i \leq n - 1)$;

$$\mu = E(U) = \sum_{i=1}^{n-1} E(X_i) = (n-1)\frac{a}{n}\frac{b}{n-1} = \frac{ab}{n}.$$

Also

$$\sigma_U^2 = E(U^2) - [E(U)]^2 = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} E(X_iX_j) - \mu^2,$$

and the counterpart of the table in the text is as follows.

Indices	#terms	Patterns	$E(X_iX_j)$
$j=i$	$n-1$	AB	$\frac{a}{n}\cdot\frac{b}{n-1}$
$j-i=\pm1$	$2(n-2)$	impossible	0
$\lvert j-i\rvert\geq 2$	$(n-2)(n-3)$	AB,AB	$\frac{a}{n}\cdot\frac{b}{n-1}\cdot\frac{a-1}{n-2}\cdot\frac{b-1}{n-3}$
$1\leq i,j\leq n-1$	$(n-1)^2$		

Therefore, $E(U^2)=(n-1)\dfrac{a}{n}\cdot\dfrac{b}{n-1}+(n-2)(n-3)\dfrac{a}{n}\cdot\dfrac{b}{n-1}\cdot\dfrac{a-1}{n-2}\cdot\dfrac{b-1}{n-3}$, whence, by algebra, $\sigma_U^2=\dfrac{\mu(\mu+c)}{n-1}$

13. (a) Mean = 7; standard deviation ≈ 1.80. The observed number of juxtapositions (3) is more than two standard deviations away from the mean.

    (b) $p_0+p_1+p_2+p_3\approx .02506$. Here $p_0=0$.

    (c) $p_{11}+p_{12}+p_{13}+p_{14}\approx .02506$. Here $p_{14}=0$.

    (d) Consequently the P value is about 0.051.

14. (a) Mean = 5; standard deviation ≈ 1.54.

    (b) There are 8 WC's and CW's, and $p_8+p_9+p_{10}\approx 0.04956$. The evidence supports the statistician's suspicion (barely) at the 5% level of significance.

15.

$$\mu_U=E(U)=\frac{ab}{n-1};$$

$$\sigma_U^2=\frac{\mu_U(\mu_U+c-1)}{n-2}.$$

16.

$$\mu_T=E(T)=\frac{2ab}{n-1};$$

$$\sigma_T^2=\frac{\mu_T(\mu_T+c-2)}{n-2}.$$

17. Let $q(j;a,b,c)$ denote the probability of exactly j "AB" juxtapositions in a random linear permutation of a A's, b B's, and c C's, and let $q^*(j;a,b,c)$ denote the corresponding probability for a linear permutation bent into a ring. There will be j "AB" juxtapositions in the ring if the line starts with "B" and ends with "A" and has $j-1$ "AB" juxtapositions or if it does not start with "B" and end with "A" and has j "AB" juxtapositions. Thus,

$$q^*(j;a,b,c) = \frac{b}{n}\frac{a}{n-1}q(j-1;a-1,b-1,c)+q(j;a,b,c)-\frac{b}{n}\frac{a}{n-1}q(j;a-1,b-1,c)$$

18. Adapt the procedure used to find p_j. Count carefully: It is easy to go astray! For each i between 1 and $\min\{a,b\}$, count the linear arrangements of a A's and b B's yielding i circular runs of each type. There are four pattern cases ($A^+B^+\cdots A^+B^+$, etc.), and the number of circular runs, which equals the number of juxtapositions, must be even ($2i$). For each arrangement, count the ways of choosing j AB and BA contacts to keep (which sometimes involves the "wraparound" case), place a C in the other AB and BA gaps (sometimes a left or right end), and then, regarding these other gaps and the gaps among the A's and B's as urns, count the ways to place the remaining C's in these urns. Sum over i and multiply by $a!b!c!/n!$, where $n=a+b+c$, to get the desired probability, p^*_j.

    Consider, e.g., the $A^+B^+\cdots A^+B^+$ pattern with i A^+'s and i B^+'s. With a A's and b B's, this can be arranged in $\binom{a-1}{i-1}\binom{b-1}{i-1}$ ways. If the wraparound contact is maintained, then there are $j-1$ interior contacts to keep—$\binom{2i-1}{j-1}$ choices—and $2i-1-(j-1)$ contacts to break up with one C each. The remaining $r=c-(2i-j)$ C's may then be distributed into $u=a+b-j$ urns in $\binom{r+u-1}{u-1}$ ways. If the wraparound contact is broken up, then choose the j contacts to be maintained from the $2i-1$ interior possibilities—$\binom{2i-1}{j}$ ways—and place a C in each of the other $2i-1-j$ interior gaps. At least one of the remaining C's must go at one end, so first count the ways to put $r=c-(2i-1-j)$ balls into $u_1=a+b+1-j$ urns (this includes both ends), and then subtract the number of ways to put the r balls into the $u_2=a+b-1-j$ non-end urns. The total for the two cases is given by Y_{ij} below.

    After the remaining three patterns are tallied, the result is:

$$p_j^* = \sum_{i=1}^{\min\{a,b\}} \left[2\binom{a-1}{i-1}\binom{b-1}{i-1} Y_{ij} + \{ \binom{a-1}{i}\binom{b-1}{i-1} + \binom{a-1}{i-1}\binom{b-1}{i} \} Z_{ij} \right] \frac{a!b!c!}{n!}$$

where

$$Y_{ij} = \binom{2i-1}{j-1}\binom{n-2i-1}{a+b-j-1} + \binom{2i-1}{j}\left[\binom{n-2i+1}{a+b-j} - \binom{n-2i-1}{a+b-j-2}\right]$$

and

$$Z_{ij} = \binom{2i}{j}\binom{n-2i}{a+b-j}.$$

19. Mean = 2; variance = 12/11; $p_0^* = 59/924 \approx 0.06$.

20. (1)

j	0	1	2	3	4	5	6	7
p_j^*	.5065	.3752	.1037	.0137	.0009	3×10^{-5}	4×10^{-7}	2×10^{-9}

(2)

j	2	3	4	5	6	7	8	9	10	11
p_j^*	.01	.04	.09	.15	.19	.19	.15	.10	.05	.02

(3)

j	0	1	2	3	4	5	6	7	8
p_j^*	.092	.267	.322	.212	.083	.120	.003	.000	.000

(4)

j	0	1	2	3
q_j^*	$5/21 \approx .2381$	$15/28 \approx .5357$	$3/14 \approx .2143$	$1/84 \approx .0119$

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