CARRIES, COMBINATORICS, AND AN AMAZING MATRIX

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This is the story of a serendipitous discovery. It began when I was investigating a mundane subject: carries in addition. To my surprise, a probabilistic perspective and some heavy-duty number crunching revealed a mathematical cache: an infinite collection of stochastic matrices in every dimension exhibiting an unusual symmetry and multifaceted combinatorial features. For each matrix $\mathbf{\Pi}$:

- The eigenvalues are all positive and form a finite, decreasing, geometric sequence; furthermore, if we diagonalize $\mathbf{\Pi}$ as $\mathbf{U}^{-1}\mathbf{\Pi}\mathbf{U} = \mathbf{D}$, where the eigenvalues are arranged in decreasing order in the diagonal matrix $\mathbf{D}$, then, aside from a constant of proportionality:
- The entries in the row of $\mathbf{U}^{-\mathbf{1}}$ corresponding to the eigenvalue 1 are Eulerian numbers.
- The entries in the row of $\mathbf{U}^{-\mathbf{1}}$ corresponding to the least eigenvalue are the entries in a row of Pascal's triangle, but with alternating signs; the entries in the column of $\mathbf{U}$ corresponding to this eigenvalue are their reciprocals.
- The entries in the first and last rows of $\mathbf{U}$ are respectively unsigned and signed Stirling numbers of the first kind.

These unanticipated relationships first came to light when I explored the territory numerically, using Mathematica®. That started me on a project that cycled through phases of computer experimentation, conjecture, and rigorous mathematics. The mathematics involved included generating functions, recurrence relations, summation and matrix manipulation, combinatorial identities, and discrete probability—the techniques of "concrete mathematics" ([8]; see also [18]). This article is an invitation to aficionados of concrete mathematics to enjoy a guided tour of some wonderful sights. Along the way we will also point out several interesting side trips (exercises) for explorers.

THE PROBLEM. When we add two long random base-ten (say) numbers, how often do we have a carry (of 1) from one column to the next? For example, consider the following addition of two fifty-digit numbers composed of digits taken from a table of random numbers:

010011	00110	11100	01111	00001	00000	01101	11111	00000	1100
24003	80475	19793	71578	52010	72216	15692	96689	80452	46312
+16129	49245	21693	20946	60874	82351	32516	23823	30046	06870
40133	29720	41486	92525	12885	54567	48209	20513	10498	53182

We observe that we got a carry-out of 0 in 27 cases and a carry-out of 1 in 23 cases, or 54% and 46%, respectively. It would be natural to conjecture that in the long run, as the number of digits increases without bound, the relative frequencies would be 50%–50%. This is true, and a thorough treatment is given in [12, pp. 262–263].

What happens if we add *three* long random numbers? When I asked some faculty colleagues, they conjectured that carries of 0, 1, and 2 would be equally likely. In a seminar for students, one participant confidently asserted that there would be mostly 1's. In the following sum of three 50-digit random numbers

111011	10111	11000	10111	10210	11102	11122	01011	11210	2112
05453	03060	83621	43443	07082	04401	15299	64642	73497	38426
67711	70528	46700	00171	55077	11440	95932	91116	17255	19649
76306	39287	31026	49339	70267	68885	98147	70311	43856	37376
149471	12876	61347	92954	32426	84728	09380	26070	34608	95451

we have 12 (24%) carries of 0, 31 (62%) carries of 1, and 7 (14%) carries of 2; perhaps the student was right.

But are these empirical percentages good estimates of the long-run frequencies? And more generally, what is the long-run frequency of each possible carry value when we add any number of long numbers represented in any base?

THE CARRIES PROCESS. Consider the addition of m random n-digit base-b numbers:

$$\begin{array}{llllllll}
\text{Carries} & C_n & C_{n-1} & C_{n-2} & \cdots & C_2 & C_1 & C_0 = 0 \\
\text{Addends} & & X_{1,n-1} & X_{1,n-2} & \cdots & X_{1,2} & X_{1,1} & X_{1,0} \\
 & & \vdots & \vdots & & \vdots & \vdots & \vdots \\
 & + & X_{m,n-1} & X_{m,n-2} & \cdots & X_{m,2} & X_{m,1} & X_{m,0} \\
\hline
\text{Sum} & S_n & S_{n-1} & S_{n-2} & \cdots & S_2 & S_1 & S_0
\end{array}$$

We assume that the $\{X_{h,k}\}$ are independent uniformly distributed random digits. The key to our analysis is this probabilistic insight: *the* *carries form a finite Markov chain*:

$$\Pr(C_{k+1} = c_{k+1} \mid C_k = c_k, \ldots, C_1 = c_1, C_0 = 0) = \Pr(C_{k+1} = c_{k+1} \mid C_k = c_k).$$

This is true because the carry-out C_{k+1} depends only on C_k and, of course, the digits $X_{1,k}, X_{2,k}, \ldots,$ and $X_{m,k}$.

What are the possible values of C_k? Those who have experience with adding long columns of figures by hand know that the carry-out can be anything from 0 to $m-1$.[1] Thus the state space of the carries process $\langle C_k \rangle$ is $\{0, 1, \ldots, m-1\}$. Furthermore, it is possible to get from any state to any other state in $\lfloor \log_b(m-1) \rfloor + 1$ steps. A probabilist would say that this Markov chain is acyclic (aperiodic) and irreducible.

Let $\mathbf{\Pi} = [\pi_{ij}]$ denote the transition matrix:

$$\pi_{ij} = \Pr(\text{carry-out} = j \mid \text{carry-in} = i) \quad \text{where} \quad 0 \le i, j \le m-1.$$

Because the states of the Markov chain are numbered $0, \ldots, m-1$, we number the rows and columns of $\mathbf{\Pi}$ in the same way. Now, to calculate π_{ij}, consider the base-b addition in the k^{th} place:

$$C_{k+1} = j \iff jb \le i + X_{1,k} + \cdots + X_{m,k} < (j+1)b$$

where $0 \le X_{1,k}, \ldots, X_{m,k} \le b-1$. Introducing the slack variable Y, we observe that this is equivalent to

$$X_{1,k} + \cdots + X_{m,k} + Y = (j+1)b - 1 - i =: z \tag{1}$$

where $0 \le X_{1,k}, \ldots, X_{m,k}, Y \le b-1$. As Tucker [16, p. 311] notes concerning a similar problem, "By using generating functions to solve this problem, we [do] not need to know anything about the inclusion-exclusion complexities of this problem. Generating functions automatically [perform] the required combinatorial logic!" So now we invoke generating functions (and we gear up to the level of chapter 6 of [16] or chapter 2 of *generatingfunctionology* [18]). The number of integer solutions of (1) is the same as the coefficient of x^z in $(1 + x + x^2 + \cdots + x^{b-1})^{m+1}$. Because

$$(1 + x + x^2 + \cdots + x^{b-1})^{m+1} = (1 - x^b)^{m+1}(1-x)^{-(m+1)}$$

and

$$(1 - x^b)^{m+1} = \sum_r \binom{m+1}{r} (-x^b)^r$$

and

$$(1 - x)^{-(m+1)} = \sum_{s \ge 0} \binom{m+s}{m} x^s,$$

[1] An interesting induction problem is to prove that the maximum possible value of the carry C_k is $m - 1 - \lfloor (m-1)/b^k \rfloor$.

the desired coefficient is

$$\sum_{r\le z/b}(-1)^r\binom{m+1}{r}\binom{m+z-rb}{m}.$$

Since $r \le z/b = j+1-(i+1)/b$ if and only if $r \le j - \lfloor i/b \rfloor$, we may summarize our result as follows.

Theorem 1. *The carries process* $\langle C_k \rangle$ *for the base-*b *addition of* m *random numbers is a finite Markov chain with state space* $\{0,1,\dots,m-1\}$ *and transition matrix* $\mathbf{\Pi} = [\pi_{ij}]$ *given by*

$$\pi_{ij} = b^{-m}\sum_{r=0}^{j-\lfloor i/b\rfloor}(-1)^r\binom{m+1}{r}\binom{m-1-i+(j+1-r)b}{m}.$$

When $b = 2$, the number of bit-valued solutions of (1) is simply $\binom{m+1}{z}$, so

$$\pi_{ij} = 2^{-m}\binom{m+1}{2j-i+1} \qquad \text{in the binary case.}$$

Let's look at some other examples. When $b = 10$ and $m = 2,3,4$, then $\mathbf{\Pi}$ is

$$\begin{bmatrix}0.55 & 0.45\\ 0.45 & 0.55\end{bmatrix}, \qquad \begin{bmatrix}0.220 & 0.660 & 0.120\\ 0.165 & 0.670 & 0.165\\ 0.120 & 0.660 & 0.220\end{bmatrix}, \qquad \begin{bmatrix}0.0715 & 0.5280 & 0.3795 & 0.0210\\ 0.0495 & 0.4840 & 0.4335 & 0.0330\\ 0.0330 & 0.4335 & 0.4840 & 0.0495\\ 0.0210 & 0.3795 & 0.5280 & 0.0715\end{bmatrix}.$$

The 0.0210 in the upper right corner, for example, signifies that, given a carry-in of 0 to a column of 4 random decimal digits, the probability of a carry-out of 3 is 0.0210. For a general base b we obtain the following formulas when $m = 2,3$:

$$\mathbf{\Pi} = \frac{1}{2b}\begin{bmatrix}b+1 & b-1\\ b-1 & b+1\end{bmatrix} \quad \text{and} \quad \mathbf{\Pi} = \frac{1}{6b^2}\begin{bmatrix}b^2+3b+2 & 4b^2-4 & b^2-3b+2\\ b^2-1 & 4b^2+2 & b^2-1\\ b^2-3b+2 & 4b^2-4 & b^2+3b+2\end{bmatrix}.$$

CROSS-SYMMETRY. These examples reveal that $\mathbf{\Pi}$ has an unusual sort of symmetry: it is radially symmetric about its center. A typical crossword puzzle grid has the same sort of symmetry. This symmetry is familiar to matrix theorists, who call it "centrosymmetry," and to statisticians, who call it "cross-symmetry." See [17] for a survey.

Theorem 2. *For $i, j = 0, 1, \ldots, m-1$, we have $\pi_{m-1-i,m-1-j} = \pi_{i,j}$, i.e.,*

$$\Pr(C_{k+1} = m-1-j | C_k = m-1-i) = \Pr(C_{k+1} = j | C_k = i).$$

Proof: The cross-symmetry is not obvious from the formula in Theorem 1, so we turn to the probabilistic definition. Given that $C_k = i$, we have $C_{k+1} = j$ if and only if

$$jb \leq i + X_{1,k} + X_{2,k} + \cdots + X_{m,k} \leq (j+1)b - 1. \tag{2}$$

The $\{X_{h,k}\}$ are independent random variables that are uniformly distributed on $\{0, 1, \ldots, b-1\}$. Accordingly, the equation $\tilde{X}_{h,k} := b - 1 - X_{h,k}$ defines independent random variables that are also uniformly distributed on $\{0, 1, \ldots, b-1\}$. Now if we negate the inequalities (2) and add $mb - 1$, we get

$$(m-1-j+1)b-1 \geq m-1-i+\tilde{X}_{1,k}+\tilde{X}_{2,k}+\cdots+\tilde{X}_{m,k} \geq (m-1-j)b,$$

which is the condition for $C_{k+1} = m-1-j$ given that $C_k = m-1-i$. ■

EIGENVALUES AND EIGENVECTORS AND SERENDIPITY. Let's return to the carries problem. It is well known in Markov chain theory that our original question concerning the long-run relative frequencies of the carry values is answered by the stationary probability vector, i.e., the row vector $\mathbf{v} = (\mathbf{p_0}, \ldots, \mathbf{p_{m-1}})$ with nonnegative entries summing to 1 that satisfies $\mathbf{v\Pi} = \mathbf{v}$. Thus, $\mathbf{v}$ is the left eigenvector of $\mathbf{\Pi}$ associated with the eigenvalue 1. When I used Mathematica® to calculate some sample cases, out of curiosity I asked for more than $\mathbf{v}$ alone; I asked for the entire eigensystem. That's when I discovered the surprises hidden in the matrix $\mathbf{\Pi}$.

Let's look at the eigenvalues first. For $b = 10$ and $m = 2, 3, 4, 5$, we find these eigenvalue sets: $\{1, 0.1\}$, $\{1, 0.1, 0.01\}$, $\{1, 0.1, 0.01, 0.001\}$, and $\{1, 0.1, 0.01, 0.001, 0.0001\}$. For $b = 2$ and $m = 5$ we get $\{1, 1/2, 1/4, 1/8, 1/16\}$.

Conjecture 1: The eigenvalues of $\mathbf{\Pi}$ are given by the geometric sequence $1, b^{-1}, \ldots, b^{-(m-1)}$.

The eigenvectors for the two $m = 5$ cases ($b = 10$ and $b = 2$) turn out to be the same. Further numerical experimentation shows that the eigenvectors are independent of the base!

Conjecture 2: The eigenvectors do *not* depend on b.

What do these eigenvectors look like? If we assemble these (row) eigenvectors in a matrix $\mathbf{V} = [\mathbf{v_{ij}}] = [\mathbf{v_{ij}(m)}]$ for $m = 2, 3, 4$, and 5, we get:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 11 & 11 & 1 \\ 1 & 3 & -3 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 26 & 66 & 26 & 1 \\ 1 & 10 & 0 & -10 & -1 \\ 1 & 2 & -6 & 2 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \tag{3}$$

Familiar sequences emerge at the bottom and top of $\mathbf{V}$.

Conjecture 3: The bottom row of $\mathbf{V}$ is proportional to a row of Pascal's triangle, but with alternating signs.

Conjecture 4: The top row of $\mathbf{V}$ is proportional to a row of Eulerian numbers.

EULERIAN NUMBERS. The first few Eulerian numbers are listed in the following table.

n	$\left\langle {n \atop 0} \right\rangle$	$\left\langle {n \atop 1} \right\rangle$	$\left\langle {n \atop 2} \right\rangle$	$\left\langle {n \atop 3} \right\rangle$	$\left\langle {n \atop 4} \right\rangle$	$\left\langle {n \atop 5} \right\rangle$	$\left\langle {n \atop 6} \right\rangle$
0	1						
1	1	0					
2	1	1	0				
3	1	4	1	0			
4	1	11	11	1	0		
5	1	26	66	26	1	0	

It appears that $v_{0j}(m) = \left\langle {m \atop j} \right\rangle$ for $j = 0, \dots, m-1$. The Eulerian numbers, first discussed by Euler (of course) in [6, pp. 485–487], [7, pp. 373–375], arise naturally in the study of random permutations; see [13, sect. 5.1.3] and the references there, [2], [4], [5, ch. 10], [14], and [15]. They satisfy the recurrence relation (see [8, sect. 6.2])

$$\left\langle {n \atop k} \right\rangle = (k+1) \left\langle {n-1 \atop k} \right\rangle + (n-k) \left\langle {n-1 \atop k-1} \right\rangle \qquad \text{for integer } n > 0 \tag{4}$$

with the boundary condition $\left\langle {0 \atop k} \right\rangle = \delta_{0k}$, the Kronecker delta. Using this relation and induction, one may deduce (as in [3])

$$\sum_k \left\langle {n \atop k} \right\rangle = n!. \tag{5}$$

Anticipating the verification of Conjecture 4, we normalize the Eulerian numbers in accordance with (5) to get *the stationary probabilities for the carries process*:

$$p_j = \frac{1}{m!}\left\langle {m \atop j} \right\rangle \qquad \text{for } j = 0, \dots m-1.$$

In particular, the long-run relative frequencies of carry values are $(\frac{1}{2}, \frac{1}{2})$ for $m = 2$ and $(\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$ for $m = 3$. We see that our empirical values came reasonably close.

The explicit formula

$$\left\langle {n \atop k} \right\rangle = \sum_{r=0}^{k} (-1)^r \binom{n+1}{r} (k+1-r)^n \tag{6}$$

was given by Euler himself. Notation for Eulerian numbers is not standardized; our notation conforms to that of [8].

THE EULERIAN RECURRENCE AND V. How can we find an explicit formula for $\mathbf{V}$, the matrix whose rows are the left eigenvectors of $\mathbf{\Pi}$? If we are clever or lucky, we can guess the right answer and then verify it.

Playing with the $\mathbf{V}$ cases in (3), we find that the Eulerian recurrence (4) holds also for *every* row of $\mathbf{V}$, i.e.,

$$v_{ij}(m) = (j+1)v_{ij}(m-1) + (m-j)v_{i,j-1}(m-1) \qquad \text{for } 0 \le i < m, \tag{7}$$

where we define $v_{i,-1}(m) = 0$ and $v_{im}(m) = 0$. This recurrence cannot give us the last row of $\mathbf{V}(m)$ in terms of $\mathbf{V}(m-1)$, because the latter matrix is short one row. But if Conjecture 3 is correct, we can paste in the last row of $\mathbf{V}(m)$ by the formula

$$v_{m-1,j}(m) = (-1)^j \binom{m-1}{j}.$$

A little calculation shows that $\left| {m \atop j} \right| := (-1)^j \binom{m-1}{j}$ has the near-Eulerian property

$$\left| {m \atop j} \right| = (j+1) \left| {m \atop j} \right| + (m-j) \left| {m \atop j-1} \right|,$$

so (7) will be satisfied for $i = m-1$ if we define $v_{m-1,j}(m-1) = (-1)^j \binom{m-1}{j}$, or,

$$v_{mj}(m) = (-1)^j \binom{m}{j}. \tag{8}$$

This is an equation for the row *below* the bottom row of $\mathbf{V}$. Now, to see the pattern that generalizes, we make the nonobvious observation [1, p. 822] that

$$(-1)^j\binom{m}{j} = \binom{m+1}{0} - \binom{m+1}{1} + \cdots + (-1)^j\binom{m+1}{j}.$$

This equation, (8), Conjecture 4, and (6) give

$$v_{mj}(m) = \sum_{r=0}^{j}(-1)^r\binom{m+1}{r}(j+1-r)^0 \;\&\; v_{0j}(m) = \sum_{r=0}^{j}(-1)^r\binom{m+1}{r}(j+1-r)^m,$$

so we conjecture that

$$v_{ij} = v_{ij}(m) = \sum_{r=0}^{j}(-1)^r\binom{m+1}{r}(j+1-r)^{m-i}. \qquad (9)$$

Theorem 3. *Let* $\mathbf{V} = [\mathbf{v_{ij}}]$ *be the* $m \times m$ *matrix given by (9) for* $0 \le i, j < m$*, and let* $\mathbf{D} = diag\{1, b^{-1}, \ldots, b^{-m+1}\}$*. Then*

$$\mathbf{V\Pi V^{-1}} = \mathbf{D}.$$

Assuming Theorem 3, we have $\mathbf{\Pi} = \mathbf{V^{-1}DV}$; this is an equation that may be used to define $\mathbf{\Pi} = \mathbf{\Pi}(b)$ for every complex $b \neq 0$ and to prove $\mathbf{\Pi}(ab) = \mathbf{\Pi}(a)\mathbf{\Pi}(b)$ for all nonzero complex numbers a and b. When a and b are bases—say $a = 5$ and $b = 2$, whence $ab = 10$—this may be explained as follows. We may rewrite each base-ten digit T in the mixed-radix system having bases 5 and 2: $T = 0\times5\times2 + F\times2 + B$; now when the carry-in i is applied to the binary column, it leads to an intermediate carry of k to the base-5 column with probability $\pi_{ik}(2)$, which then generates a carry-out of j with probability $\pi_{kj}(5)$, and so $\pi_{ij}(10) = \sum \pi_{ik}(2)\pi_{kj}(5)$, i.e., $\mathbf{\Pi}(10) = \mathbf{\Pi}(2)\mathbf{\Pi}(5)$.

PROOF: CONCRETE MATHEMATICS AHEAD. Here we'll make heavy use of "concrete mathematics" techniques. First we observe that

$$v_{ij} = \sum_{r=0}^{j+1}(-1)^r\binom{m+1}{r}(j+1-r)^{m-i}$$

is the convolution, or Cauchy product, of the sequences (in k) $\left\langle(-1)^k\binom{m+1}{k}\right\rangle$ and $\langle k^{m-i}\rangle$ evaluated at $j+1$, i.e.,

$$v_{ij} = \text{coefficient of } x^{j+1} \text{ in } \sum_{k\ge0}(-1)^k\binom{m+1}{k}x^k \cdot \sum_{k\ge0}k^{m-i}x^k.$$

Now we use the binomial theorem and the generating function of $\langle k^n \rangle$,

$$\sum_{k\geq 0} k^n x^k = (x\frac{d}{dx})^n (1-x)^{-1}, \tag{10}$$

to get

$$\begin{aligned} v_{ij} &= \text{coefficient of } x^{j+1} \text{ in } (1-x)^{m+1}(x\frac{d}{dx})^{m-i}(1-x)^{-1} \\ &= \text{coefficient of } x^j \text{ in } x^{-1}(1-x)^{m+1}(x\frac{d}{dx})^{m-i}(1-x)^{-1}. \end{aligned}$$

Thus we obtain the generating function of the i^{th} row of $\mathbf{V}$:

$$\sum_{j\geq 0} v_{ij} x^j = x^{-1}(1-x)^{m+1}(x\frac{d}{dx})^{m-i}(1-x)^{-1}. \tag{11}$$

When $i = 0$, this generating function is $x^{-1} f_m(x)$ where $f_m(x)$ is the Eulerian polynomial of degree m (see [14], [4]).

We must show

$$\sum_{k=0}^{m-1} v_{ik}\pi_{kj} = b^{-i} v_{ij} \text{ for } i, j = 0, 1, \ldots, m-1.$$

By substituting, interchanging the order of summation (an entertaining exercise!), and simplifying, we get

$$\begin{aligned} \sum_{k=0}^{m-1} v_{ik}\pi_{kj} &= b^{-m} \sum_{k=0}^{m-1} \sum_{r=0}^{j-\lfloor k/b \rfloor} (-1)^r \binom{m+1}{r} \binom{m-1-k+(j+1-r)b}{m} v_{ik} \\ &= b^{-m} \sum_{r=0}^{j} \sum_{k=0}^{(m-1)\wedge((j+1-r)b-1)} (-1)^r \binom{m+1}{r} \binom{m-1-k+(j+1-r)b}{m} v_{ik} \\ &= b^{-m} \sum_{r=0}^{j} (-1)^r \binom{m+1}{r} \sum_{k=0}^{(j+1-r)b-1} \binom{m-1-k+(j+1-r)b}{m} v_{ik}. \end{aligned}$$

Let $K = (j+1-r)b-1$. The inner sum, $\sum_{k=0}^{K} \binom{m+K-k}{m} v_{ik}$, is the convolution of the sequences (in k) $\left\langle \binom{m+k}{m} \right\rangle$ and $\langle v_{ik} \rangle$ evaluated at K. We know that

$$\sum_{k\geq 0} \binom{m+k}{m} x^k = (1-x)^{-m-1},$$

and we have the generating function of $\langle v_{ik} \rangle$ in (11). Thus, the inner sum is equal to the coefficient of x^K in

$$(1-x)^{-m-1} \cdot x^{-1}(1-x)^{m+1}(x\frac{d}{dx})^{m-i}(1-x)^{-1} = x^{-1}(x\frac{d}{dx})^{m-i}(1-x)^{-1},$$

which, invoking (10), is

$$\sum_{k=0}^{K} \binom{m+K-k}{m} v_{ik} = (K+1)^{m-i} = ((j+1-r)b)^{m-i}.$$

Therefore,

$$b^{-m} \sum_{r=0}^{j} (-1)^r \binom{m+1}{r} \sum_{k=0}^{K} \binom{m+K-k}{m} v_{ik} = b^{-m} \sum_{r=0}^{j} (-1)^r \binom{m+1}{r} ((j+1-r)b)^{m-i}$$
$$= b^{-i} \sum_{r=0}^{j} (-1)^r \binom{m+1}{r} (j+1-r)^{m-i} = b^{-i} v_{ij}.$$

■

CONFIRMATION OF THE CONJECTURES AND MORE. Theorem 3 tells us that the rows of $\mathbf{V}$ are left eigenvectors of $\mathbf{\Pi}$ corresponding to the eigenvalues $1, b^{-1}, \ldots, b^{-(m-1)}$, so Conjectures 1 and 2 are true. Also the formula for v_{0j} is the same as the explicit formula (6) for the Eulerian numbers, so Conjecture 4 is true. Finally, letting $i = m-1$ in (11), we get

$$\sum_{j \geq 0} v_{m-1,j} x^j = x^{-1}(1-x)^{m+1} x \frac{d}{dx} (1-x)^{-1} = (1-x)^{m-1},$$

so Conjecture 3 follows, by the binomial theorem.

There are other patterns in $\mathbf{V}$. It is easy to verify that the leftmost column of $\mathbf{V}$ is all 1's. It is a little harder to show that the rightmost column has alternating $+1$'s and -1's, but it is a splendid opportunity to use the calculus of finite differences. Both exercises are left to the reader.

THE RIGHT EIGENVECTOR MATRIX U: EMPIRICAL RESULTS. Let's look at the right eigenvectors of the transition matrix $\mathbf{\Pi}$. As an alternative to direct computation of the eigenvectors, we may compute the inverse of the matrix $\mathbf{V}$. Numerical experimentation reveals that, in order to get integer values, we should multiply by $m!$, so we let $\mathbf{U} = m!\mathbf{V}^{-1}$. For $m = 2, 3, 4, 5$, we find that $\mathbf{U}$ is:

$$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \quad \begin{bmatrix}1 & 3 & 2\\ 1 & 0 & -1\\ 1 & -3 & 2\end{bmatrix}, \quad \begin{bmatrix}1 & 6 & 11 & 6\\ 1 & 2 & -1 & -2\\ 1 & -2 & -1 & 2\\ 1 & -6 & 11 & -6\end{bmatrix}, \quad \begin{bmatrix}1 & 10 & 35 & 50 & 24\\ 1 & 5 & 5 & -5 & -6\\ 1 & 0 & -5 & 0 & 4\\ 1 & -5 & 5 & 5 & -6\\ 1 & -10 & 35 & -50 & 24\end{bmatrix}.$$

Tantalizing patterns are already visible in these first few examples. Even though the columns are the eigenvectors, the top and bottom *rows* leap out at the combinatorial *cognoscenti*: They are Stirling numbers of the first kind! The pattern of the eigenvector in the last column may be exposed by dividing by $(m-1)!$: reciprocals of binomial coefficients with alternating signs! Forming difference tables of the columns reveals more patterns: It appears that the j^{th} difference of the j^{th} column is a constant—$(-1)^j m!/(m-j)!$—which would make the j^{th} column a polynomial of degree j in the row index i. To summarize, for the matrix $\mathbf{U} = m!\mathbf{V^{-1}}$, we propose:

Conjecture 5: Column j is a degree-j polynomial function of row index i.

Conjecture 6: The entries in the final column are proportional to reciprocals of entries in a row of Pascal's triangle with alternating signs.

Conjecture 7: The top row consists of unsigned Stirling numbers of the first kind (in reverse order).

Conjecture 8: The bottom row consists of signed Stirling numbers of the first kind (in reverse order).

STIRLING NUMBERS OF THE FIRST KIND. The first few (unsigned) Stirling numbers of the first kind are as follows.

n	$\left[{n \atop 0}\right]$	$\left[{n \atop 1}\right]$	$\left[{n \atop 2}\right]$	$\left[{n \atop 3}\right]$	$\left[{n \atop 4}\right]$	$\left[{n \atop 5}\right]$	$\left[{n \atop 6}\right]$
0	1						
1	0	1					
2	0	1	1				
3	0	2	3	1			
4	0	6	11	6	1		
5	0	24	50	35	10	1	

The Stirling number $\left[{n \atop k}\right]$ may be characterized combinatorially as the number of ways n objects can be arranged into k cycles, but for our purposes we characterize Stirling numbers algebraically. Rising factorial powers may be represented in terms of ordinary powers by means of unsigned Stirling numbers of the first kind:

$$x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1) = \sum_k \left[\begin{array}{c} n \\ k \end{array} \right] x^k; \qquad (12)$$

falling factorial powers may be represented in terms of ordinary powers by means of signed Stirling numbers the first kind:

$$x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1) = \sum_k (-1)^{n-k} \left[\begin{array}{c} n \\ k \end{array} \right] x^k. \qquad (13)$$

(See [8, sect. 6.1] [11, pp. 65–68], or [10, ch. 4].)

THE RIGHT EIGENVECTORS. How can we find an explicit formula for the matrix $\mathbf{U}$ of right eigenvectors of $\mathbf{\Pi}$? One way would be to solve $\mathbf{\Pi U} = \mathbf{UD}$, which appears to be very difficult. My way was to find $\mathbf{U}$ by solving $\mathbf{UV} = \mathbf{m!I}$. It turns out to take longer to solve this equation than it does to prove the answer is right, so let's start with the answer.

Theorem 4. *Let* $\mathbf{V} = [\mathbf{v_{ij}}]$ *be the* $m \times m$ *matrix given by (9). Then* $m!\mathbf{V^{-1}} = [\mathbf{u_{ij}}]$ *where*

$$u_{ij} = u_{ij}(m) = \sum_{r=m-j}^{m} (-1)^{m-r} \left[\begin{array}{c} m \\ r \end{array} \right] \binom{r}{m-j} (m-1-i)^{r-(m-j)}$$

for $0 \le i, j < m$ *and where* 0^0 *is taken to be* 1.

Proof: We shall show that $\sum_{k=0}^{m-1} u_{ik} v_{kj} = m!\delta_{ij}$. We start with the standard trick of interchanging the order of summation:

$$\begin{aligned} \sum_{k=0}^{m-1} u_{ik} v_{kj} &= \sum_{k=0}^{m-1} u_{ik} \sum_{r=0}^{j} (-1)^r \binom{m+1}{r} (j+1-r)^{m-k} \\ &= \sum_{r=0}^{j} (-1)^r \binom{m+1}{r} \sum_{k=0}^{m-1} u_{ik} (j+1-r)^{m-k}. \end{aligned}$$

Here we rewrite the inner sum as follows (note that $\left[\begin{matrix} m \\ 0 \end{matrix}\right] = 0$ in the second line and the interchange trick is used again in the third line):

$$\begin{aligned}
\sum_{k=0}^{m-1} u_{ik}(j+1-r)^{m-k} &= \sum_{k=1}^{m} u_{i,m-k}(j+1-r)^{k} \\
&= \sum_{k=0}^{m}\sum_{s=k}^{m}(-1)^{m-s}\left[\begin{matrix} m \\ s \end{matrix}\right]\binom{s}{k}(m-1-i)^{s-k}(j+1-r)^{k} \\
&= \sum_{s=0}^{m}(-1)^{m-s}\left[\begin{matrix} m \\ s \end{matrix}\right]\sum_{k=0}^{s}\binom{s}{k}(m-1-i)^{s-k}(j+1-r)^{k} \\
&= \sum_{s=0}^{m}(-1)^{m-s}\left[\begin{matrix} m \\ s \end{matrix}\right](m-1-i+j+1-r)^{s} \qquad \text{[by the binomial theorem]} \\
&= (m-i+j-r)^{\underline{m}} \qquad \text{[by (13)]} \\
&= m!\binom{m-i+j-r}{m}.
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{m-1} u_{ik}v_{kj} = m!\sum_{r=0}^{j}(-1)^{r}\binom{m+1}{r}\binom{m-i+j-r}{m}.$$

Note that $\binom{m-i+j-r}{m} = 0$ if $0 \le m-i+j-r < m$, i.e., $j-i < r \le m-i+j$. If $0 \le j < i < m$, every term in the last equation is 0; if $0 \le i = j < m$, only the $r = 0$ term is nonzero (it is $m!$); if $0 \le i < j < m$, we may add zero terms to get

$$\sum_{r=0}^{m+1}(-1)^{r}\binom{m+1}{r}\binom{m-i+j-r}{m} = \Delta^{m+1}(\text{polynomial in } r \text{ of degree } m) = 0.$$

Therefore,

$$\sum_{k=0}^{m-1} u_{ik}v_{kj} = m!\delta_{ij}.$$

■

CONFIRMATION OF THE CONJECTURES. The formula for **U** is complicated enough that it still takes some work to verify our conjectures. Conjectures 5, 6, and 8 are left as an exercises, and we turn to Conjecture 7, which claims that the top row of **U** contains unsigned Stirling numbers of the first kind: For $j = 0, \ldots, m-1$,

$$u_{0j}(m) = \left[\begin{matrix} m \\ m-j \end{matrix} \right].$$

This conjecture neatly reduces to the two basic identities relating Stirling numbers of the first kind to factorial powers. By Theorem 4, for $n = 1, \ldots, m$,

$$u_{0,m-n}(m) = \sum_{r=n}^{m} (-1)^{m-r} \left[\begin{matrix} m \\ r \end{matrix} \right] \binom{r}{n} (m-1)^{r-n}.$$

Thus, Conjecture 7 is equivalent to the identity

$$\sum_{r=n}^{m} (-1)^{m-r} \left[\begin{matrix} m \\ r \end{matrix} \right] \binom{r}{n} (m-1)^{r-n} = \left[\begin{matrix} m \\ n \end{matrix} \right]. \tag{14}$$

Fix $m \geq n$, and let a_n denote the left side of (14). Switching the order of summation, we find that the generating function of $\langle a_n \rangle$ is

$$\begin{aligned}
\sum_{n \geq 0} a_n x^n &= \sum_{r=0}^{m} (-1)^{m-r} \left[\begin{matrix} m \\ r \end{matrix} \right] \sum_{n=0}^{r} \binom{r}{n} x^n (m-1)^{r-n} \\
&= \sum_{r=0}^{m} (-1)^{m-r} \left[\begin{matrix} m \\ r \end{matrix} \right] (x+m-1)^r \qquad \text{[by the binomial theorem]} \\
&= (x+m-1)^{\underline{m}} \qquad \text{[by (13)]} \\
&= (x+m-1)(x+m-2)\cdots(x) \\
&= x^{\overline{m}},
\end{aligned}$$

which is the generating function of $\left\langle \left[\begin{matrix} m \\ n \end{matrix} \right] \right\rangle$, by (12). Therefore, $a_n = \left[\begin{matrix} m \\ n \end{matrix} \right]$, i.e., identity (14) holds, so Conjecture 7 is true.

FURTHER CONSEQUENCES AND EXPLORATIONS. Many people are fascinated by combinatorial identities like (14), and there are many to be found in the context of the carries transition matrix. For example, the empirical recurrence identity for $\mathbf{V}$, (7), is indeed true, and provides a family of arrays satisfying the Eulerian recurrence (4). Other identities, including familiar ones, may be extracted from the matrix equations $\mathbf{\Pi U} = \mathbf{UD}$, $\mathbf{V\Pi} = \mathbf{DV}$, $\mathbf{UV} = \mathbf{m!I}$, and $\mathbf{VU} = \mathbf{m!I}$. Here is just one illustration: Set $i = 0$ and $j = m-1 > 0$

in $\sum_k v_{ik}u_{kj} = m!\delta_{ij}$ and get

$$\sum_{k=0}^{m-1}(-1)^k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \bigg/ \binom{m-1}{k} = 0.$$

Besides identities, we have seen geometric sequences, binomial coefficients, Eulerian numbers, and Stirling numbers of the first kind. What about other special numbers, like Stirling numbers of the second kind? Are they lurking nearby? Yes, indeed. Stirling numbers of the second kind crop up naturally in formulas for the factorial moments of the stationary probability distribution of $\mathbf{\Pi}$; alternatively, the n^{th} factorial moment is exactly the generalized Bernoulli number $B_n^{(n-m)}$ (see [5, chapter 15]). A different sort of result is that the stationary probabilities are asymptotically normally distributed [5, pp. 150–154]. I hope some readers are inspired to discover other interesting connections.

Going beyond these kinds of propositions, we may put the matrix $\mathbf{\Pi}$ in a larger context: It is the $x = 1$ case of the matrix $[\pi_{ij}x^j]$, which plays a central role in the analysis of the asymptotic prime-power divisibility of multinomial coefficients [9].

Our tour has revealed a combinatorial richness hidden in the matrix $\mathbf{\Pi}$. But it leaves unanswered the question, *Why* are all these combinatorially significant relationships connected with the carries matrix?

ACKNOWLEDGMENTS. I thank Jennifer Galovich and Paul Fjelstad for pointing out the connections with Eulerian numbers, and I thank Paul Fjelstad, Stephen Hilding, Ron Rietz, and Herbert Wilf for their comments on earlier drafts of this paper.

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