

**FRACTAL DIMENSION OF GENERALIZED BINOMIAL
COEFFICIENTS MODULO A PRIME:
PRELIMINARY REPORT**

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ABSTRACT. Given a sequence (u_n) of positive integers generated by $u_1 = 1, u_2 = a, u_n = au_{n-1} + bu_{n-2} (n \geq 3)$, define the generalized factorial by $[n]! = u_1 u_2 \cdots u_n$ and the generalized binomial coefficient by $C(i, j) = [i+j]! / ([i]![j]!)$. Assume that the prime p does not divide b . Let $r = \min\{n : p|u_n\}$. **Theorem 1 (Asymptotic abundance of residues):** $\#\{(i, j) | 0 \leq i, j < rp^k \text{ and } C(i, j) \equiv \rho \pmod{p}\} \sim \frac{r(r+1)}{2(p-1)} \binom{p+1}{2}^k$ as $k \rightarrow \infty$ for $\rho = 1, \dots, p-1$. **Theorem 2 (Fractal dimension):** Let $s_k = rp^k$. The Hausdorff dimension of $\bigcap_k \bigcup_{i,j < s_k} \{[i/s_k, (i+1)/s_k) \times [j/s_k, (j+1)/s_k) : p \nmid C(i, j)\}$ is $\log \binom{p+1}{2} / \log p$.

LUCAS'S THEOREM AND PASCAL'S TRIANGLE MODULO p

A remarkable theorem of E. Lucas [15] expresses the binomial coefficient $\binom{N}{m}$ modulo a prime p in terms of the binomial coefficients of the base- p digits of N and m : If $N = \sum N_j p^j$ and $m = \sum m_j p^j$ where $0 \leq N_j, m_j < p$, then

$$\binom{N}{m} \equiv \prod \binom{N_j}{m_j} \pmod{p}.$$

This paper exploits a generalization of the following alternative version of this theorem: Let

$$B(m, n) = \binom{m+n}{m} = \frac{(m+n)!}{m!n!};$$

then

$$B(m, n) \equiv B(m \div p, n \div p) B(m \bmod p, n \bmod p) \pmod{p}$$

where $m \div p$ is the integer quotient of m by p , and $m \bmod p$ is the remainder.

This theorem implies that the residues of Pascal's triangle modulo p have a self-similar structure; see, e.g., [17], [2], [6], [7], [14], [22],

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and [1]. For example, if $p = 3$, then the matrix $[B(m, n) \bmod p]$ for $0 \leq m, n < 9$ is given as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1\mathbf{B} & 1\mathbf{B} & 1\mathbf{B} \\ 1\mathbf{B} & 2\mathbf{B} & 0\mathbf{B} \\ 1\mathbf{B} & 0\mathbf{B} & 0\mathbf{B} \end{bmatrix} \pmod{p},$$

where

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \pmod{p},$$

so this matrix is the tensor (or, Kronecker) product $\mathbf{B} \otimes \mathbf{B} \bmod p$. Generally, as noted in [16], modulo p we have that $[B(m, n) \bmod p]$ for $0 \leq m, n < p^k$ will be $\mathbf{B}^{\otimes k}$, the k -fold tensor product of $\mathbf{B} = [B(i, j) \bmod p]$ where $0 \leq i, j < p$. Note that matrix indices start at index pair $(0, 0)$.

Wolfram [22] and Flath and Peele [4] have determined that the “fractal” dimension of geometric representations of the pattern of the nonzero residues of Pascal’s triangle is given by $\log \binom{p+1}{2} / \log p$. The purpose of this paper is to prove similar fractal dimension results, as well as density results, for a large class of generalized binomial coefficients.

GENERALIZED BINOMIAL COEFFICIENTS

Generalized binomial coefficients corresponding to a given sequence (u_n) are defined analogously to $B(m, n)$ by replacing $n!$ by the product of u_1 through u_n ,

$$[n]! := \prod_{j=1}^n u_j,$$

and then defining

$$C(m, n) = \frac{[m+n]!}{[m]![n]}$$

(assuming any zero factors in the numerator and denominator are first paired and then cancelled).

In this paper we assume that the sequence is defined by a second-order recurrence relation as follows:

$$u_0 = 0; u_1 = 1; u_n = au_{n-1} + bu_{n-2} \text{ for } n = 2, 3, 4, \dots$$

where a and b are integers.

When $a = 2$ and $b = -1$, then $u_n = n$ and the generalized binomial coefficients become the ordinary binomial coefficients: $C(m, n) = B(m, n)$. When $a = 1 + q$ and $b = -q$, then $u_n = 1 + q + q^2 + \dots + q^{n-1}$ and the generalized binomial coefficients are the Gauss q -binomial coefficients. When $a = 1$ and $b = 1$, then $u_n = F_n$, the n^{th} Fibonacci number, and the generalized binomial coefficients become the fibonomial coefficients.

WELLS'S THEOREM AND THE PATTERN OF THE RESIDUES

Wells [18] [19] has proved a generalization of the Lucas theorem for these generalized binomial coefficients. For the purposes of our fractal dimension calculations, we use one of the alternative versions given in [10]. To state it, we need to introduce the following definitions and notations.

Definition 1. Let r denote the rank of apparition of p ; thus, $r = \min\{n \in \mathbb{N} : u_n \equiv 0 \pmod{p}\}$. Let t denote the (least) period of $\langle u_n \pmod{p} \rangle$, if it exists. Let $s = t/r$.

Notation: If $r < \infty$, then for each nonnegative integer n , let

$$\begin{aligned} n_0 &= n \pmod{r}, \\ n' &= n \div r, \\ n^* &= n \pmod{t}, \\ n'' &= n^* \div r = n' \pmod{s}. \end{aligned}$$

Definition 2. For $i, j \geq 0$ and for $0 \leq k, l < r$, let $A_{i,j}(k, l)$ denote the solution of the modulo- p recurrence relation

$$A_{i,j}(k, l) \equiv u_{ir+k+1}A_{i,j}(k, l-1) + bu_{jr+l-1}A_{i,j}(k-1, l)$$

for $0 \leq k, l < r$ together with the boundary conditions

$$A_{i,j}(k, -1) \equiv 0 \pmod{p} \quad \text{for } 1 \leq k < r$$

and

$$A_{i,j}(-1, l) \equiv 0 \pmod{p} \quad \text{for } 1 \leq l < r$$

and

$$A_{i,j}(0, 0) \equiv 1 \pmod{p}.$$

Definition 3. If $r < \infty$, then for $i, j \geq 0$ and $0 \leq k, l < r$, define

$$H_{i,j}(k, l) = u_{r+1}^{r ij} A_{i,j}(k, l).$$

As noted in [10], $H_{m',n'}(m_0, n_0) \equiv H_{m'',n''}(m_0, n_0) \pmod{p}$, and $H_{m'',n''}(m_0, n_0) \equiv 0 \pmod{p}$ if $m_0 + n_0 < r$.

Here is the generalization of Lucas's theorem that we shall use.

Proposition 1. *If $p \nmid b$, then, for $m, n \geq 0$,*

$$C(m, n) \equiv B(m', n') H_{m'', n''}(m_0, n_0) \pmod{p}.$$

This result simplifies nicely when $s = 1$. Then $m'' = n'' = 0$, and $H_{0,0}(m_0, n_0) \equiv C(m_0, n_0) \pmod{p}$ for $0 \leq m_0, n_0 < r$. Thus, in this case, as in the Pascal "triangle" case, the pattern of residues exhibits self-similarity upon scaling by p .

Corollary. *If $p \nmid b$ and $s = 1$, then, for $m, n \geq 0$,*

$$C(m, n) \equiv B(m', n') C(m_0, n_0) \pmod{p},$$

or, letting \mathbf{B} denote the matrix $[B(i, j)]$ with $0 \leq i, j < p$ and $\mathbf{C}_k = [C(m, n)]$ with $0 \leq m, n < rp^k$, we have

$$\mathbf{C}_k \equiv \mathbf{B}^{\otimes k} \otimes \mathbf{C}_0 \pmod{p}.$$

The following examples are borrowed from [10].

Example 1: q -binomial coefficients. Take $u_n = \sum_{k=0}^{n-1} q^k$ to obtain the q -binomial coefficients. For a numerical example, take $q = 2$ and $p = 5$. Then $u_1 = 1, u_2 = 3, u_3 = 7, u_4 = 15, u_5 = 31, \dots$, whence $r = 4$, and

$$\mathbf{C}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 7 & 15 \\ 1 & 7 & 35 & 155 \\ 1 & 15 & 155 & 1395 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \pmod{5},$$

so

$$\mathbf{C}_1 \equiv \mathbf{B} \otimes \mathbf{C}_0 \equiv \begin{bmatrix} 1\mathbf{C}_0 & 1\mathbf{C}_0 & 1\mathbf{C}_0 & 1\mathbf{C}_0 & 1\mathbf{C}_0 \\ 1\mathbf{C}_0 & 2\mathbf{C}_0 & 3\mathbf{C}_0 & 4\mathbf{C}_0 & 0\mathbf{C}_0 \\ 1\mathbf{C}_0 & 3\mathbf{C}_0 & 1\mathbf{C}_0 & 0\mathbf{C}_0 & 0\mathbf{C}_0 \\ 1\mathbf{C}_0 & 4\mathbf{C}_0 & 0\mathbf{C}_0 & 0\mathbf{C}_0 & 0\mathbf{C}_0 \\ 1\mathbf{C}_0 & 0\mathbf{C}_0 & 0\mathbf{C}_0 & 0\mathbf{C}_0 & 0\mathbf{C}_0 \end{bmatrix} \pmod{5}.$$

Example 2: Fibonomial coefficients modulo p . Let $a = b = 1$ so that $u_n = F_n$, and consider the case $p = 3$. Then $r = 4$, $t = 8$, and $s = 2$. The initial part of the table of fibonomial coefficients modulo 3 is given in Table 1.

1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1
1 1 2 0	2 2 1 0	1 1 2 0	2 2 1 0	1 1 2 0	2 2 1 0	1 1 2 0	2 2 1 0	1 1 2 0
1 2 0 0	1 2 0 0	1 2 0 0	1 2 0 0	1 2 0 0	1 2 0 0	1 2 0 0	1 2 0 0	1 2 0 0
1 0 0 0	2 0 0 0	1 0 0 0	2 0 0 0	1 0 0 0	2 0 0 0	1 0 0 0	2 0 0 0	1 0 0 0
1 2 1 2	2 1 2 1	0 0 0 0	1 2 1 2	2 1 2 1	0 0 0 0	1 2 1 2	2 1 2 1	0 0 0 0
1 2 2 0	1 2 2 0	0 0 0 0	2 1 1 0	2 1 1 0	0 0 0 0	1 2 2 0	1 2 2 0	0 0 0 0
1 1 0 0	2 2 0 0	0 0 0 0	1 1 0 0	2 2 0 0	0 0 0 0	1 1 0 0	2 2 0 0	0 0 0 0
1 0 0 0	1 0 0 0	0 0 0 0	2 0 0 0	2 0 0 0	0 0 0 0	1 0 0 0	1 0 0 0	0 0 0 0
1 1 1 1	0 0 0 0	0 0 0 0	1 1 1 1	0 0 0 0	0 0 0 0	1 1 1 1	0 0 0 0	0 0 0 0
1 1 2 0	0 0 0 0	0 0 0 0	2 2 1 0	0 0 0 0	0 0 0 0	1 1 2 0	0 0 0 0	0 0 0 0
1 2 0 0	0 0 0 0	0 0 0 0	1 2 0 0	0 0 0 0	0 0 0 0	1 2 0 0	0 0 0 0	0 0 0 0
1 0 0 0	0 0 0 0	0 0 0 0	2 0 0 0	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0
1 2 1 2	1 2 1 2	1 2 1 2	2 1 2 1	2 1 2 1	2 1 2 1	0 0 0 0	0 0 0 0	0 0 0 0
1 2 2 0	2 1 1 0	1 2 2 0	1 2 2 0	2 1 1 0	1 2 2 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 0 0	1 1 0 0	1 1 0 0	2 2 0 0	2 2 0 0	2 2 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 0 0 0	2 0 0 0	1 0 0 0	1 0 0 0	2 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 1 1	2 2 2 2	0 0 0 0	2 2 2 2	1 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 2 0	1 1 2 0	0 0 0 0	1 1 2 0	1 1 2 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 0 0	2 1 0 0	0 0 0 0	2 1 0 0	1 2 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 0 0 0	1 0 0 0	0 0 0 0	1 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 1 2	0 0 0 0	0 0 0 0	2 1 2 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 2 0	0 0 0 0	0 0 0 0	1 2 2 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 0 0	0 0 0 0	0 0 0 0	2 2 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 0 0 0	0 0 0 0	0 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 1 1	1 1 1 1	1 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 2 0	2 2 1 0	1 1 2 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 0 0	1 2 0 0	1 2 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 0 0 0	2 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 1 2	2 1 2 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 2 0	1 2 2 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 0 0	2 2 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 0 0 0	1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 2 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 2 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0

TABLE 1. The Fibonomial coefficients modulo 3

$$\begin{bmatrix} 1\mathbf{H}_{0,0} & 1\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & 1\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & 1\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & 1\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & \cdots \\ 1\mathbf{H}_{1,0} & 2\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 1\mathbf{H}_{1,1} & 2\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 1\mathbf{H}_{1,0} & 2\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & \cdots \\ 1\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 1\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & \cdots \\ 1\mathbf{H}_{1,0} & 1\mathbf{H}_{1,1} & 1\mathbf{H}_{1,0} & 2\mathbf{H}_{1,1} & 2\mathbf{H}_{1,0} & 2\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & \cdots \\ 1\mathbf{H}_{0,0} & 2\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 2\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & \cdots \\ 1\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 2\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & \cdots \\ 1\mathbf{H}_{0,0} & 1\mathbf{H}_{0,1} & 1\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & \cdots \\ 1\mathbf{H}_{1,0} & 2\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & 0\mathbf{H}_{1,1} & 0\mathbf{H}_{1,0} & \cdots \\ 1\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & 0\mathbf{H}_{0,1} & 0\mathbf{H}_{0,0} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

TABLE 2. Submatrices of the fibonomial coefficients mod 3

By Definition 3,

$$\mathbf{H}_{0,0} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \mathbf{H}_{0,1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}; \mathbf{H}_{1,0} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \mathbf{H}_{1,1} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}.$$

The structure of the matrix of fibonomial coefficients modulo 3, in accordance with Proposition 1, is given in Table 2. Wells [21] provides a detailed description of the pattern of these submatrices from a “triangular” perspective. The pattern of fibonomial coefficients modulo any prime is given in [8].

Proposition 1 and the example show that the infinite matrix $[C(i, j) \bmod p]$ may be partitioned into $r \times r$ submatrices which form basic, natural “tiling units.” The pattern of the residues is obtained by superimposing the self-similar array of binomial coefficients modulo p upon the doubly periodic “tiling” of the plane by “hidden” $r \times r$ \mathbf{H} matrices. The binomial structure is self-similar upon scaling by the factor p . The $r \times r$ tiling structure has period s both horizontally and vertically, and so the period is t at the element level. When $s = 1$, there are $p - 1$ different nonzero $r \times r$ submatrices, one for each nonzero residue value of $B(m', n') \bmod p$ times \mathbf{C}_0 . In the general case, there are also $s \cdot s$ different $H_{m'', n''}$ -matrices. In fact, there are $(p - 1)s^2$ different nonzero “tiles” [10].

Proposition 2. *Assume $p \nmid b$ and $r < \infty$. The number of different nonzero r -by- r submatrices of the infinite matrix $[C(i, j) \pmod p]$ is $(p-1)s^2$.*

In the case of the the fibonomial coefficients modulo 3, the exhibited matrix shows these seven submatrices:

$$1\mathbf{H}_{0,0}, 1\mathbf{H}_{0,1}, 1\mathbf{H}_{1,0}, 1\mathbf{H}_{1,1}, 2\mathbf{H}_{0,1}, 2\mathbf{H}_{1,0}, 2\mathbf{H}_{1,1}.$$

The missing case, $2\mathbf{H}_{0,0}$, must be sought farther out. The places of the missing $2\mathbf{H}_{0,0}$ are $(5, 11), (11, 5), (5, 13), (13, 5) \dots$ in Table 2.

SCALING-UP RECURSION FORMULA

Define

$$C_{\alpha,\beta}(m, n) \equiv B(m', n')H_{\alpha+m'', \beta+n''}(m_0, n_0) \pmod p.$$

Using Proposition 1 and the fact that $s|p-1$, we may deduce the following recursion relation.

Proposition 3. *If $m = m_k p^{k-1} r + m^{(k)}$ and $n = n_k p^{k-1} r + n^{(k)}$ where $0 \leq m^{(k)}, n^{(k)} < r p^{k-1}$, then*

$$C_{\alpha,\beta}(m, n) \equiv B(m_k, n_k)C_{m_k+\alpha, n_k+\beta}(m^{(k)}, n^{(k)}) \pmod p.$$

ASYMPTOTIC ABUNDANCE OF RESIDUES

Also define

$$\begin{aligned} \mathfrak{C}_{\alpha,\beta}^{(k)} &= [C_{\alpha,\beta}(m, n)] \quad (0 \leq m, n < r p^k), \\ f_{\alpha,\beta}^{(k)}(\rho, \mu, \nu) &= \#\{(i, j) : 0 \leq i, j < p^k, C_{\alpha,\beta}(ir + i_0, jr + j_0) \equiv \rho H_{\mu,\nu}(i_0, j_0) \text{ for } 0 \leq i_0, j_0 < r\}, \\ f^{(k)}(\rho, \mu, \nu) &= f_{0,0}^{(k)}(\rho, \mu, \nu). \end{aligned}$$

The quantity $f^{(k)}(\rho, \mu, \nu)$ is our focus for now. It is the number of $\rho\mathbb{H}_{\mu\nu}$ tiles in the initial $r p^k \times r p^k$ square of $C(i, j)$ values.

Lemma 1.

$$f_{\alpha,\beta}^{(k)}(\rho, \mu, \nu) = f^{(k)}(\rho, \mu - \alpha, \nu - \beta).$$

Lemma 2. *For $1 \leq \rho < p$ and $0 \leq \mu, \nu < s$,*

$$f^{(k)}(\rho, \mu, \nu) = \sum_{\substack{0 \leq i, j < p \\ i+j < p}} f^{(k-1)}(B(i, j)^{-1} \rho, \mu - i, \nu - j).$$

Letting $I = (\rho, \mu, \nu)$ and $J = (\tilde{\rho}, \tilde{\mu}, \tilde{\nu})$, define

$$Q_J^I = \#\{(i, j) : 0 \leq i, j < p, i + j < p, B(i, j)^{-1}\rho \equiv \tilde{\rho} \pmod{p}, \\ \mu - i \equiv \tilde{\mu} \pmod{p}, \quad \nu - j \equiv \tilde{\nu} \pmod{p}\}$$

and

$$\mathbb{Q} = [Q_J^I].$$

Lemma 3.

$$\sum_J Q_J^I = p(p+1)/2 \quad \text{and} \quad \sum_I Q_J^I = p(p+1)/2.$$

Accordingly, let

$$P_J^I = \frac{2}{p(p+1)} Q_J^I.$$

Then

$$\mathbb{P} := [P_J^I]$$

is a doubly stochastic matrix.

Lemma 4. *Every entry of \mathbb{Q}^3 is positive, and so is every entry of \mathbb{P}^3 . Consequently, the matrix \mathbb{Q} is primitive, and the finite Markov chain having \mathbb{P} as its transition matrix is regular.*

Lemma 5. *Regarding $f^{(k)}$ as a row vector with indices $I = (\rho, \mu, \nu)$, we have*

$$f^{(k)} = f^{(0)} \mathbb{Q}^k.$$

Theorem 1. *For $1 \leq \rho < p$ and $0 \leq \mu, \nu < s$,*

$$f^{(n)}(\rho, \mu, \nu) \sim \frac{1}{(p-1)s^2} \left[\frac{p(p+1)}{2} \right]^n \quad \text{as } n \rightarrow \infty$$

and

$$\lim_{n \rightarrow \infty} \frac{\log f^{(n)}(\rho, \mu, \nu)}{\log p^n} = \frac{\log p(p+1)/2}{\log p}.$$

For $\rho = 0$ we have that

$$\sum_{\mu, \nu} f^{(n)}(\rho, \mu, \nu) = p^{2n} - [p(p+1)/2]^n = p^{2n} [1 - \{(p+1)/(2p)\}^n]$$

is the number of zero tiles in \mathbb{C}_n

Proof. The first part of the theorem follows from the previous lemmas by applying Perron-Frobenius theory. The last statement follows from Kummer's theorem as follows. According to Kummer's theorem, the number of pairs (i, j) with $0 \leq i, j < p^n$ for which p does not divide the binomial coefficient $B(i, j)$ is the same as the number of pairs of n -digit p -ary numbers $(\sum_{k=0}^{n-1} i_k p^k, \sum_{k=0}^{n-1} j_k p^k)$ for which there are no carries

when added in base- p arithmetic. This is the same as the number of digit pairs (i_k, j_k) with $i_k + j_k < p$, which is $[p(p+1)/2]^n$. Therefore the number of nonzero tiles $B(i, j)\rho H_{\mu, \nu}$ in \mathbb{C}_n is precisely $[p(p+1)/2]^n$, and the number of zero tiles is $p^{2n} - [p(p+1)/2]^n$. \square

Corollary 1. *Let*

$$R_n(\rho) = \#\{(i, j) : 0 \leq i, j < rp^n, C(i, j) \equiv \rho \pmod{p}\},$$

the number of $C(i, j)$'s in the initial $rp^n \times rp^n$ square congruent to ρ modulo p . Then the asymptotic abundance of the residue ρ , where $1 \leq \rho < p$, is given by

$$R_n(\rho) \sim \frac{r(r+1)}{2(p-1)} \left[\frac{p(p+1)}{2} \right]^n,$$

and so the logarithmic density, or box-counting dimension, of the set of generalized binomial coefficients that are congruent to ρ is

$$\lim_{n \rightarrow \infty} \frac{\log R_n(\rho)}{\log p^n} = \frac{\log[p(p+1)/2]}{\log p}.$$

Proof. Let

$$g(\rho, \mu, \nu) = \#\{(i, j) : 0 \leq i, j < r, H_{\mu, \nu}(i, j) \equiv \rho \pmod{p}\},$$

the number of entries in the $r \times r$ matrix $H_{\mu, \nu}$ that are congruent to ρ modulo p . Then

$$R_n(\rho) = \sum_{1 \leq \tilde{\rho} < p} \sum_{0 \leq \mu, \nu < s} f^{(n)}(\rho \tilde{\rho}^{-1}, \mu, \nu) g(\tilde{\rho}, \mu, \nu),$$

so

$$\begin{aligned} R_n(\rho) &\sim \sum_{1 \leq \tilde{\rho} < p} \sum_{0 \leq \mu, \nu < s} \frac{[p(p+1)/2]^n}{(p-1)s^2} g(\tilde{\rho}, \mu, \nu) \\ &= \frac{[p(p+1)/2]^n}{(p-1)s^2} \sum_{0 \leq \mu, \nu < s} \sum_{1 \leq \tilde{\rho} < p} g(\tilde{\rho}, \mu, \nu) \\ &= \frac{[p(p+1)/2]^n}{(p-1)s^2} \cdot s^2 \cdot \frac{r(r+1)}{2}. \quad \square \end{aligned}$$

HAUSDORFF DIMENSION OF $C(m, n) \pmod{p}$

A ‘‘fractal set’’ corresponding to the pattern of all nonzero residues of the generalized binomial coefficients modulo a prime p is constructed as a subset of the square $[0, 1) \times [0, 1)$ by ‘‘tremas’’ as follows. Flath and Peele [4] give an alternative, rescaled lattice construction.

For each k let \mathcal{G}_k denote the class of sets

$$G_{m,n}^{(k)} = \bigcup_{\substack{0 \leq i, j < r \\ i+j < r}} \left[\frac{mr+i}{rp^k}, \frac{mr+i+1}{rp^k} \right) \times \left[\frac{nr+j}{rp^k}, \frac{nr+j+1}{rp^k} \right)$$

with $0 \leq m, n < p^k$ and $p \nmid B(m, n)$, and let G_k be their union. Proposition 1 and Lucas's theorem imply that $G_{m,n}^{(k)}$ is contained in some set in \mathcal{G}_{k-1} and contains a finite number of disjoint sets of \mathcal{G}_{k+1} , and $G_{k+1} \subset G_k$. Accordingly our fractal set is

$$G := \bigcap_{k \in \mathbb{N}} G_k.$$

Theorem 2. *If p is a prime that does not divide b , then the fractal set G constructed above has Hausdorff dimension*

$$\dim_H(G) = \frac{\log \binom{p+1}{2}}{\log p}.$$

Proof. The proof uses (1) the fact that

$$\dim_H G \leq \dim_B G,$$

where $\dim_B G$ is the box-counting dimension of G , and (2) the mass distribution principle [3, p. 55]: if μ is a measure on a set F and for some s there are numbers $c > 0, \delta > 0$ such that

$$\mu(U) \leq c|U|^s$$

for all sets U with $|U| \leq \delta$ (where $|U|$ is the diameter of U), then the Hausdorff measure $\mathcal{H}^s(F) \geq \mu(F)/c$ and $s \leq \dim_H(F)$. The box-counting dimension of G may be calculated [3, p. 41] by the formula

$$\dim_B G = \lim_{k \rightarrow \infty} \frac{\log N_{\delta_k}(G)}{-\log \delta_k},$$

where $N_{\delta}(G)$ is the smallest number of δ -mesh squares that intersect the set G , provided that the sequence (δ_k) decreases to zero and $\delta_{k+1} \geq \eta \delta_k$ for some positive constant η . Let us choose $\delta_k = 1/(rp^k)$ (and $\eta = 1/p$). Then

$$N_{\delta_k}(G) = \frac{r(r+1)}{2} \left[\frac{p(p+1)}{2} \right]^k,$$

so

$$\begin{aligned}
\dim_B(G) &= \lim_{k \rightarrow \infty} \frac{\log N_{\delta_k}(G)}{-\log \delta_k} \\
&= \lim_{k \rightarrow \infty} \frac{\log \left(\frac{r(r+1)}{2} \left[\frac{p(p+1)}{2} \right]^k \right)}{-\log[1/(rp^k)]} \\
&= \lim_{k \rightarrow \infty} \frac{\log \frac{r(r+1)}{2} + k \log \frac{p(p+1)}{2}}{\log r + k \log p} \\
&= \frac{\log \binom{p+1}{2}}{\log p}.
\end{aligned}$$

Now let μ be the “natural measure” defined by repeated subdivision [3, pp. 13–14] that assigns weight $\binom{p+1}{2}^{-k}$ to each set in \mathcal{G}_k and weight 0 to the complement of G_k . (At the next stage, the weight of each $G_{m,n}^{(k)}$ is evenly divided among the $\binom{p+1}{2}$ sets in \mathcal{G}_{k+1} contained therein.) We shall see that there exist $c > 0$ and $\delta > 0$ such that

$$\mu(U) \leq c|U|^d \quad \text{where} \quad d := \frac{\log \binom{p+1}{2}}{\log p}$$

for all sets U with diameter $|U| \leq \delta$. Let $\delta \in (0, 1)$. Suppose $|U| \leq \delta$. Let k be the integer such that $1/p^{k+1} \leq |U| < 1/p^k$. Note that then $1/p^k \leq p|U|$ and U meets at most four of the sets in \mathcal{G}_k (because U is contained in a square of side $|U|$ with sides parallel the coordinate axes, and this containing square can intersect no more than four $G_{m,n}^{(k)}$'s). Therefore,

$$\begin{aligned}
\mu(U) &\leq 4 \frac{1}{\binom{p+1}{2}^k} \\
&= \frac{4}{(p^d)^k} \quad \left[\text{because } d = \frac{\log \binom{p+1}{2}}{\log p} \right] \\
&= 4 \left(\frac{1}{p^k} \right)^d \\
&\leq 4(p|U|)^d,
\end{aligned}$$

so $\mu(U) \leq c|U|^d$ for all sets U with $|U| \leq \delta$ where $c = 4p^d$. By the mass distribution principle, $d \leq \dim_H G$. But from before, $\dim_B G = d$, and we know $\dim_H G \leq \dim_B G$, so we must have $\dim_H G = d = \log \binom{p+1}{2} / \log p$. \square

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