

# CARRIES, COMBINATORICS, AND AN AMAZING MATRIX

EARLY, UNPOLISHED VERSION OF *MONTHLY* ARTICLE  
WITH ADDITIONAL RESULTS AND PROOFS

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This is the story of a serendipitous discovery. It began with an excavation in the drab landscape of grade school arithmetic. The effort was facilitated and made more interesting by employing the power tools of probability theory. The end result was the uncovering of a treasure-trove of mathematical gems: an infinite collection of stochastic matrices in every dimension exhibiting an unusual symmetry and multifaceted combinatorial features: For each matrix  $\mathbf{\Pi}$ :

- The eigenvalues form a (finite, decreasing) geometric sequence; furthermore, if the similarity transformation diagonalizing  $\mathbf{\Pi}$  is written  $\mathbf{U}^{-1}\mathbf{\Pi}\mathbf{U} = \mathbf{D}$ , where the eigenvalues are arranged in decreasing order in the diagonal matrix  $\mathbf{D}$ , then:
- The row of  $\mathbf{U}^{-1}$  corresponding to the eigenvalue 1 is proportional to the Eulerian numbers.
- The row of  $\mathbf{U}^{-1}$  corresponding to the least eigenvalue is proportional to a row of Pascal's triangle but with alternating signs, and the column of  $\mathbf{U}$  corresponding to the same eigenvalue is proportional to the reciprocals.
- The top and bottom rows of  $\mathbf{U}$  are proportional to the unsigned and signed Stirling numbers of the first kind.

These jewels first came to light when I explored the territory numerically, using Mathematica<sup>®</sup>. That started me on a project that repeatedly cycled through phases of computer experimentation, conjecture, and rigorous mathematics. The mathematics involved included generating functions, recurrence relations, summation and matrix manipulation, combinatorial identities, and discrete probability—the techniques of “concrete mathematics” ([9]; see also [20]). This article is an invitation to aficionados of concrete mathematics to enjoy a guided tour of some wonderful sights.

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## THE PROBLEM

When we add two long random base-ten (say) numbers, how often do we have a carry (of 1) from one column to the next? For example, consider the following addition of two fifty-digit numbers composed of digits taken from a table of random numbers:

010011	00110	11100	01111	00001	00000	01101	11111	00000	1100
24003	80475	19793	71578	52010	72216	15692	96689	80452	46312
+16129	49245	21693	20946	60874	82351	32516	23823	30046	06870
40133	29720	41486	92525	12885	54567	48209	20513	10498	53182

We observe that we got a carry-out of 0 in 27 cases and a carry-out of 1 in 23 cases, or 54% and 46%, respectively. It would be natural to conjecture that in the long run, as the number of digits increases without bound, the relative frequencies would be 50%–50%. In fact this is true. (A thorough treatment of this case is given in [13, pp. 262–263].)

What will happen if we add *three* long random numbers? When I asked some faculty, they tentatively conjectured that carries of 0, 1, and 2 would be uniformly distributed; in a seminar for students, one participant confidently asserted that there would be a lot of 1's. Let's look at an example, a sum of three 50-digit random numbers:

111011	10111	11000	10111	10210	11102	11122	01011	11210	2112
05453	03060	83621	43443	07082	04401	15299	64642	73497	38426
67711	70528	46700	00171	55077	11440	95932	91116	17255	19649
76306	39287	31026	49339	70267	68885	98147	70311	43856	37376
149471	12876	61347	92954	32426	84728	09380	26070	34608	95451

We see that we have 12 (24%) carries of 0, 31 (62%) carries of 1, and 7 (14%) carries of 2; it seems the student was right.

To get a sense of the behavior of carry frequencies for longer columns, let's do one more example, the addition of four random 50-digit numbers:

112221	11112	31112	12112	21122	12111	11222	23211	21221	2212
54746	71115	78218	64314	11227	41702	54517	97676	14078	45317
56819	27340	07200	52663	57864	85159	15460	97564	29637	27742
34990	62122	38223	28526	37006	22774	46026	15981	87291	56946
02269	22795	87593	81830	95383	67823	20196	54850	46779	64519
148825	83374	11236	27335	01482	17459	36201	66072	77786	94524

Here we get a carry of 0 in no cases (0%), a carry of 1 in 26 cases (52%), a carry of 2 in 22 cases (44%), and a carry of 3 in 2 cases (4%)—the

carry values seem to be concentrating around an average of 1.5. But are these empirical percentages good estimates of the long-run frequencies?

More generally, what is the long-run frequency of each possible carry value when we add any number  $m$  of long  $n$ -digit numbers represented in any base  $b$ ? The time has come to abandon our empirical investigations for a much more powerful and efficient theoretical approach.

### THE CARRIES PROCESS

Consider the addition of  $m$  random  $n$ -digit base- $b$  numbers:

Carries	$C_n$	$C_{n-1}$	$C_{n-2}$	$\cdots$	$C_2$	$C_1$	$C_0 = 0$
Addends		$X_{1,n-1}$	$X_{1,n-2}$	$\cdots$	$X_{1,2}$	$X_{1,1}$	$X_{1,0}$
		$\cdot$	$\cdot$		$\cdot$	$\cdot$	$\cdot$
		$\cdot$	$\cdot$		$\cdot$	$\cdot$	$\cdot$
		$\cdot$	$\cdot$		$\cdot$	$\cdot$	$\cdot$
	+	$X_{m,n-1}$	$X_{m,n-2}$	$\cdots$	$X_{m,2}$	$X_{m,1}$	$X_{m,0}$
Sum		$S_n$	$S_{n-1}$	$S_{n-2}$	$\cdots$	$S_2$	$S_1$
				$\cdots$	$S_2$	$S_1$	$S_0$

We assume that the  $\{X_{h,k}\}$  are independent (uniform) random digits, and we let  $n \rightarrow \infty$ . The key to our analysis is this probabilistic insight: *the carries form a finite Markov chain*:

$$\Pr(C_{k+1} = c_{k+1} \mid C_k = c_k, \dots, C_1 = c_1, C_0 = 0) = \Pr(C_{k+1} = c_{k+1} \mid C_k = c_k).$$

This is true because the carry-out  $C_{k+1}$  depends only on  $C_k$  and  $X_{1,k}, X_{2,k}, \dots, X_{m,k}$ . (See, e.g., [8] for background on probability and Markov chains.)

What are the possible values of  $C_k$ ? Those who have experience with adding long columns of figures by hand know that the carry-out can be anything from 0 to  $m - 1$  ( $m$  is the length of the column). The others may check by induction that the maximum possible value of the carry  $C_k$  is  $m - 1 - \lfloor (m - 1)/b^k \rfloor$ , and so, for  $k > \log_b(m - 1)$ , the maximum possible carry is  $m - 1$ . Thus the state space of our Markov chain is  $\{0, 1, \dots, m - 1\}$ , and, furthermore, it is possible to get from any state to any other state in  $\lfloor \log_b(m - 1) \rfloor + 1$  steps. (A probabilist would say that this Markov chain is acyclic and irreducible.)

Fix  $b$ . Let  $\mathbf{\Pi} = [\pi_{ij}] = [\pi_{ij}(m)]$  denote the transition matrix:

$$\pi_{ij} = \Pr(\text{carry-out} = j \mid \text{carry-in} = i).$$

To calculate  $\pi_{ij}$ , consider the base- $b$  addition in the  $k^{\text{th}}$  place:

$$C_{k+1} = j \iff jb \leq i + X_{1,k} + \dots + X_{m,k} < (j + 1)b$$

where  $0 \leq X_{1,k}, \dots, X_{m,k} \leq b-1$ . Introducing the “slack” variable  $Y$ , we observe that this is equivalent to

$$X_{1,k} + \dots + X_{m,k} + Y = (j+1)b - 1 - i$$

where  $0 \leq X_{1,k}, \dots, X_{m,k}, Y \leq b-1$ . Now we invoke generating functions. (And we gear up to the level of chapter 2 of *generatingfunctionology* [20].) The number of integer solutions of these inequalities is the same as the coefficient of  $x^{(j+1)b-1-i}$  in  $(1+x+x^2+\dots+x^{b-1})^{m+1}$ . Because

$$(1+x+x^2+\dots+x^{b-1})^{m+1} = (1-x^b)^{m+1}(1-x)^{-(m+1)}$$

and

$$(1-x^b)^{m+1} = \sum_r \binom{m+1}{r} (-x^b)^r$$

and

$$(1-x)^{-(m+1)} = \sum_{s \geq 0} \binom{m+s}{m} x^s,$$

the desired coefficient is

$$\sum_{r \leq j+1-(i+1)/b} (-1)^r \binom{m+1}{r} \binom{m+(j+1)b-1-i-rb}{m}.$$

Since  $r \leq j+1-(i+1)/b$  iff  $r \leq j - \lfloor i/b \rfloor$ , we may summarize our result as follows.

**Theorem 1.** *The carries process  $\langle C_k \rangle$  is a finite Markov chain with state space  $\{0, 1, \dots, m-1\}$  and transition matrix  $\mathbf{\Pi} = [\pi_{ij}]$  where*

$$\pi_{ij} = \pi_{ij}(m) = b^{-m} \sum_{r=0}^{j-\lfloor i/b \rfloor} (-1)^r \binom{m+1}{r} \binom{m-1-i+(j+1-r)b}{m}.$$

Hmmm. At first glance, this formula does not look pretty. (Appreciation of its beauty is an acquired taste.) Let’s look at some special instances. When  $b = 10$  and  $m = 2, 3, 4$ , then  $\mathbf{\Pi}$  is

$$\begin{bmatrix} 0.55 & 0.45 \\ 0.45 & 0.55 \end{bmatrix}, \quad \begin{bmatrix} 0.220 & 0.660 & 0.120 \\ 0.165 & 0.670 & 0.165 \\ 0.120 & 0.660 & 0.220 \end{bmatrix}, \quad \begin{bmatrix} 0.0715 & 0.5280 & 0.3795 & 0.0210 \\ 0.0495 & 0.4840 & 0.4335 & 0.0330 \\ 0.0330 & 0.4335 & 0.4840 & 0.0495 \\ 0.0210 & 0.3795 & 0.5280 & 0.0715 \end{bmatrix}.$$

The 0.0210 in the upper right corner, for example, signifies that, given a carry-in of 0 to a column of  $m = 4$  random decimal digits, the probability of a carry-out of 3 is 0.0210. *Note:* Because the states of the Markov chain are numbered  $0, \dots, m-1$ , we will regard the rows

and columns as numbered in the same way. For a general base  $b$  we obtain the following formulas when  $m = 2, 3$ :

$$\mathbf{\Pi} = \frac{1}{2b} \begin{bmatrix} b+1 & b-1 \\ b-1 & b+1 \end{bmatrix} \quad \text{and} \quad \mathbf{\Pi} = \frac{1}{6b^2} \begin{bmatrix} b^2 + 3b + 2 & 4b^2 - 4 & b^2 - 3b + 2 \\ b^2 - 1 & 4b^2 + 2 & b^2 - 1 \\ b^2 - 3b + 2 & 4b^2 - 4 & b^2 + 3b + 2 \end{bmatrix}.$$

### CROSSWORD PUZZLE SYMMETRY

These examples reveal that  $\mathbf{\Pi}$  has an unusual sort of symmetry: like a crossword puzzle matrix, it is radially symmetric about its center.

**Theorem 2.** For  $i, j = 0, 1, \dots, m - 1$ , we have  $\pi_{m-1-i, m-1-j} = \pi_{i, j}$ , i.e.,

$$\Pr(C_{k+1} = m - 1 - j | C_k = m - 1 - i) = \Pr(C_{k+1} = j | C_k = i).$$

*Proof.* The crossword symmetry is not obvious from the formula in Theorem 1, so we turn to the probabilistic definition. Given that  $C_k = i$ , we have  $C_{k+1} = j$  iff

$$jb \leq i + X_{1,k} + X_{2,k} + \dots + X_{m,k} \leq (j + 1)b - 1.$$

Here the  $\{X_{h,k}\}$  are independent random variables which are uniformly distributed on  $\{0, 1, \dots, b - 1\}$ . Accordingly, the equation  $\tilde{X}_{h,k} := b - 1 - X_{h,k}$  defines independent random variables which are also uniformly distributed on  $\{0, 1, \dots, b - 1\}$ . Now if we negate the above inequalities and add  $mb - 1$ , we get

$$(m - 1 - j + 1)b - 1 \geq m - 1 - i + \tilde{X}_{1,k} + \tilde{X}_{2,k} + \dots + \tilde{X}_{m,k} \geq (m - 1 - j)b,$$

which is the condition for  $C_{k+1} = m - 1 - j$  given that  $C_k = m - 1 - i$ . The theorem now follows. ■

### ASIDE: THE SUM DIGITS

A person may ask, “Why not focus on the sum digits rather than the carries?” The answer is that they are less interesting. Let’s look back at the examples we did. Discarding the sum digit that comes from the leftmost carry out, we tabulate the digit frequencies:

Digit	0	1	2	3	4	5	6	7	8	9	$\chi^2$
$m = 2$	5	6	8	4	6	7	2	2	6	4	7.2
$m = 3$	5	4	6	4	8	3	5	5	5	5	3.2
$m = 4$	3	5	7	6	6	4	5	7	5	2	4.8

The critical  $\chi^2$  value (5% level) is 16.9; these distributions look pretty uniform.

Now, an arbitrary sum digit  $S$  in our problem arises from a carry  $C$  and random digits  $X_1, \dots, X_m$  in this way:  $S \equiv C + X_1 + \dots +$

$X_m \pmod{b}$ , where the  $X_1, \dots, X_m$  are independent random variables uniformly distributed on  $\{0, \dots, b-1\}$ . Thinking inductively, we are thus led to consider the probability distribution of  $C + U$  where  $C$  is an integer-valued random variable and  $U$  is an independent uniform random digit.

$$\begin{aligned} \Pr(C + U \equiv k \pmod{b}) &= \sum_{r=0}^{b-1} \Pr(U = r) \Pr(C \equiv k - r \pmod{b}) \\ &= \frac{1}{b} \sum_{r=0}^{b-1} \Pr(C \equiv k - r \pmod{b}) \\ &= \frac{1}{b} \cdot 1 = \frac{1}{b}. \end{aligned}$$

Therefore,  $\Pr(C + U \equiv k \pmod{b}) = 1/b$ . Remarkably, in arithmetic mod  $b$ , adding a uniform random digit to *any* independent random variable yields another uniform random digit. As a consequence,  $\Pr(C + X_1 + \dots + X_m \equiv s \pmod{b}) = 1/b$ , and, since a constant random variable is independent of any other random variable,  $\Pr(S = s) = \sum_c \Pr(C = c) \Pr(c + X_1 + \dots + X_m \equiv s \pmod{b}) = (1/b) \sum \Pr(C = c) = 1/b$ . Thus, the sum digits are all uniformly distributed random digits. Applying this result to base  $b^n$ , we get  $\Pr(S_0 = s_0, \dots, S_{n-1} = s_{n-1}) = 1/b^n = \prod \Pr(S_r = s_r)$ , and consequently the sum digits are themselves *independent* uniformly distributed random digits.

#### EIGENVALUES AND EIGENVECTORS AND SERENDIPITY

Let's return to the carries problem. It is well known in Markov chain theory that our original question concerning the long-run relative frequencies of the carry values is answered by the stationary probability vector, i.e., the probability vector  $\mathbf{v} = (\mathbf{p}_0, \dots, \mathbf{p}_{m-1})$  satisfying  $\mathbf{v}\mathbf{\Pi} = \mathbf{v}$ . Thus,  $\mathbf{v}$  is the left eigenvector of  $\mathbf{\Pi}$  associated with the eigenvalue 1. When I used Mathematica<sup>®</sup> to calculate some sample cases, out of curiosity I asked for more than  $\mathbf{v}$  alone; I asked for the entire "eigensystem." That's when I discovered the treasures hidden in the matrix  $\mathbf{\Pi}$

Let's look at the eigenvalues first. For  $b = 10$  and  $m = 2, 3, 4, 5$ , we find these eigenvalue sets:  $\{1, 0.1\}$ ,  $\{1, 0.1, 0.01\}$ ,  $\{1, 0.1, 0.01, 0.001\}$ , and  $\{1, 0.1, 0.01, 0.001, 0.0001\}$ . For  $b = 2$  and  $m = 5$  we get  $\{1, 1/2, 1/4, 1/8, 1/16\}$ .

**Conjecture 1:** The eigenvalues of  $\mathbf{\Pi}$  are given by the geometric sequence  $1, b^{-1}, \dots, b^{-(m-1)}$ .

The eigenvectors for the two  $m = 5$  cases ( $b = 10$  and  $b = 2$ ) turn out to be the same. Further numerical experimentation shows that the eigenvectors do not vary as the base varies!

**Conjecture 2:** The eigenvectors do *not* depend on  $b$ .

What do these eigenvectors look like? If we assemble these (row) eigenvectors in a matrix  $\mathbf{V} = [\mathbf{v}_{ij}] = [\mathbf{v}_{ij}(\mathbf{m})]$  for  $m = 2, 3, 4$  and  $5$ , we get:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 11 & 11 & 1 \\ 1 & 3 & -3 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 26 & 66 & 26 & 1 \\ 1 & 10 & 0 & -10 & -1 \\ 1 & 2 & -6 & 2 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \quad (1)$$

Familiar sequences emerge at the bottom and top of  $\mathbf{V}$ .

**Conjecture 3:** The bottom row of  $\mathbf{V}$  is proportional to a row of Pascal's triangle but with alternating signs.

**Conjecture 4:** The top row of  $\mathbf{V}$  is proportional to a row of Eulerian numbers.

### EULERIAN NUMBERS

The first few Eulerian numbers are listed in the following table.

$n$	$\langle n \rangle_0$	$\langle n \rangle_1$	$\langle n \rangle_2$	$\langle n \rangle_3$	$\langle n \rangle_4$	$\langle n \rangle_5$	$\langle n \rangle_6$
0	1						
1	1	0					
2	1	1	0				
3	1	4	1	0			
4	1	11	11	1	0		
5	1	26	66	26	1	0	
6	1	57	302	302	57	1	0

As noted above, it appears that  $v_{0j}(m) = \langle m \rangle_j$  for  $j = 0, \dots, m - 1$ .

The Eulerian numbers, first discussed by Euler (of course) in [6, pp. 485–487], [7, pp. 373–375], arise naturally in the study of random permutations. In one interpretation,  $\langle n \rangle_k$  is the number of permutations of  $\{1, \dots, n\}$  having  $k$  ascents (or descents), i.e.,  $k$  positions where consecutive numbers increase (or decrease). Assuming this combinatorial interpretation, one may deduce ([9, sect. 6.2]) the symmetry law,

$$\langle n \rangle_k = \langle n \rangle_{n-1-k} \quad \text{for integer } n > 0,$$

and the recurrence relation

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = (k+1) \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle + (n-k) \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle \quad \text{for integer } n > 0. \quad (2)$$

Using this relation and induction, or merely invoking the combinatorial interpretation, one may deduce (cf. [3])

$$\sum_k \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = n!. \quad (3)$$

Anticipating the verification of Conjecture 4, we normalize the Eulerian numbers in accordance with (3) to get *the stationary probabilities for the carries process*:

$$p_j = \frac{1}{m!} \left\langle \begin{matrix} m \\ j \end{matrix} \right\rangle \quad \text{for } j = 0, \dots, m-1.$$

In particular, the long-run relative frequencies of carry values are  $(\frac{1}{2}, \frac{1}{2})$  for  $m = 2$ ,  $(\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$  for  $m = 3$ , and  $(\frac{1}{24}, \frac{11}{24}, \frac{11}{24}, \frac{1}{24})$  for  $m = 4$ . We see that our empirical values came reasonably close.

Alternatively, the Eulerian numbers may be characterized by Worpitzky's identity ([21], [9, p. 269], [16]),

$$x^n = \sum_k \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \binom{x+k}{n} \quad \text{for integer } n \geq 0, \quad (4)$$

which represents a power in terms of consecutive binomial coefficients. The explicit formula,

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \sum_{r=0}^k (-1)^r \binom{n+1}{r} (k+1-r)^n, \quad (5)$$

was given by Euler himself.

It should be noted that notation for Eulerian numbers is not standardized; our notation conforms to that of [9]. Further information and applications may be found in [14, sect. 5.1.3] and the references there, [2], [4], [5, ch. 10], [15], [17], [18, p. 216], and [19, sect. 1.3].

## A TALL TALE

How can we find an explicit formula for  $\mathbf{V}$ , the matrix whose rows are the left eigenvectors of  $\mathbf{\Pi}$ ? If we are clever and lucky, we can guess the right answer and then verify it.

The following account of how the left eigenvectors were found is fictitious. The truth is that the author took a wandering path through the wilderness and then suddenly leaped to the correct answer. Since this

path is unexplainable, the following (partially) fabricated motivation for conjecturing a certain formula is presented.

Playing with the  $\mathbf{V}$  cases in (1), we find that the Eulerian recurrence (2) holds also for the rows of  $\mathbf{V}$ , i.e.,

$$v_{ij}(m) = (j + 1)v_{ij}(m - 1) + (m - j)v_{i,j-1}(m - 1) \quad \text{for } 0 \leq i < m, \quad (6)$$

where we define  $v_{i,-1}(m) = 0$  and  $v_{im}(m) = 0$ . However, as it stands, this recurrence cannot give us the last row of  $\mathbf{V}(m)$  in terms of  $\mathbf{V}(m - 1)$ , because the latter matrix is short one row. But if Conjecture 3 is correct, we can paste in the last row by the formula

$$v_{m-1,j}(m) = (-1)^j \binom{m-1}{j}.$$

Now notice (via a little calculation) that  $\left| \begin{smallmatrix} m \\ j \end{smallmatrix} \right| := (-1)^j \binom{m-1}{j}$  has the special property

$$\left| \begin{smallmatrix} m \\ j \end{smallmatrix} \right| = (j + 1) \left| \begin{smallmatrix} m \\ j \end{smallmatrix} \right| + (m - j) \left| \begin{smallmatrix} m \\ j - 1 \end{smallmatrix} \right|,$$

so (6) will be satisfied for  $i = m - 1$  if we define  $v_{m-1,j}(m - 1) = (-1)^j \binom{m-1}{j}$ , or,

$$v_{mj}(m) = (-1)^j \binom{m}{j}. \quad (7)$$

Now (6) may be cast in the attractive matrix form:

$$\begin{bmatrix} v_{00}(m) & \cdots & v_{0,m-2}(m) & v_{0,m-1}(m) \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ v_{m-2,0}(m) & \cdots & v_{m-2,m-2}(m) & v_{m-2,m-1}(m) \\ v_{m-1,0}(m) & \cdots & v_{m-1,m-2}(m) & v_{m-1,m-1}(m) \end{bmatrix} = \begin{bmatrix} v_{00}(m') & \cdots & v_{0,m-2}(m') & v_{0,m-1}(m') \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ v_{m-2,0}(m') & \cdots & v_{m-2,m-2}(m') & v_{m-2,m-1}(m') \\ v_{m-1,0}(m') & \cdots & v_{m-1,m-2}(m') & v_{m-1,m-1}(m') \end{bmatrix} \begin{bmatrix} 1 & m-1 & 0 & \cdots & 0 \\ 0 & 2 & m-2 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & m-1 & 1 \\ 0 & 0 & \cdots & 0 & m \end{bmatrix},$$

where  $m' = m - 1$ . To return to the problem at hand, at this point we reach into our toolbox and pull out from [1, p. 822] the (easily verified)

identity

$$(-1)^j \binom{m}{j} = \sum_{r=0}^j (-1)^r \binom{m+1}{r}.$$

Thus we have, by this equation and (7) plus Conjecture 4 and (5),

$$v_{mj}(m) = \sum_{r=0}^j (-1)^r \binom{m+1}{r} (j+1-r)^0 \quad \& \quad v_{0j}(m) = \sum_{r=0}^j (-1)^r \binom{m+1}{r} (j+1-r)^m,$$

so we conjecture that

$$v_{ij}(m) = \sum_{r=0}^j (-1)^r \binom{m+1}{r} (j+1-r)^{m-i}.$$

#### CONFIRMATION

This formula for  $v_{ij}(m)$  does in fact give the left eigenvectors of the transition matrix  $\mathbf{\Pi}$ .

**Theorem 3.** *Let  $\mathbf{V}$  be the  $m \times m$  matrix  $[v_{ij}]$  where*

$$v_{ij} = v_{ij}(m) = \sum_{r=0}^j (-1)^r \binom{m+1}{r} (j+1-r)^{m-i}$$

for  $0 \leq i, j < m$ , and let  $\mathbf{D} = \text{diag}\{1, b^{-1}, \dots, b^{-m+1}\}$ . Then

$$\mathbf{V}\mathbf{\Pi}\mathbf{V}^{-1} = \mathbf{D}.$$

*Proof.* Here we'll make heavy use of "concrete math" techniques. First we observe that

$$v_{ij} = \sum_{r=0}^{j+1} (-1)^r \binom{m+1}{r} (j+1-r)^{m-i}$$

is the convolution, or Cauchy product, of the sequences (in  $k$ )  $\langle (-1)^k \binom{m+1}{k} \rangle$  and  $\langle k^{m-i} \rangle$  evaluated at  $j+1$ , i.e.,

$$v_{ij} = \text{coefficient of } x^{j+1} \text{ in } \sum_{k \geq 0} (-1)^k \binom{m+1}{k} x^k \cdot \sum_{k \geq 0} k^{m-i} x^k.$$

Now we use the binomial theorem and the generating function of  $\langle k^n \rangle$ ,

$$\sum_{k \geq 0} k^n x^k = \left(x \frac{d}{dx}\right)^n (1-x)^{-1}, \quad (8)$$

to get

$$\begin{aligned} v_{ij} &= \text{coefficient of } x^{j+1} \text{ in } (1-x)^{m+1} \left(x \frac{d}{dx}\right)^{m-i} (1-x)^{-1} \\ &= \text{coefficient of } x^j \text{ in } x^{-1} (1-x)^{m+1} \left(x \frac{d}{dx}\right)^{m-i} (1-x)^{-1}. \end{aligned}$$

Thus

$$\sum_{j \geq 0} v_{ij} x^j = x^{-1} (1-x)^{m+1} \left(x \frac{d}{dx}\right)^{m-i} (1-x)^{-1}. \quad (9)$$

[*Aside:* When  $i = 0$ , this generating function is  $x^{-1} f_m(x)$  where  $f_m(x)$  is the Eulerian polynomial of degree  $m$  (see [15], [4]).]

We must show

$$\sum_{k=0}^{m-1} v_{ik} \pi_{kj} = b^{-i} v_{ij} \text{ for } i, j = 0, 1, \dots, m-1.$$

By substituting, interchanging the order of summation (an entertaining exercise!), and simplifying, we get

$$\begin{aligned} \sum_{k=0}^{m-1} v_{ik} \pi_{kj} &= b^{-m} \sum_{k=0}^{m-1} \sum_{r=0}^{j - \lfloor k/b \rfloor} (-1)^r \binom{m+1}{r} \binom{m-1-k+(j+1-r)b}{m} v_{ik} \\ &= b^{-m} \sum_{r=0}^j \sum_{k=0}^{(m-1) \wedge ((j+1-r)b-1)} (-1)^r \binom{m+1}{r} \binom{m-1-k+(j+1-r)b}{m} v_{ik} \\ &= b^{-m} \sum_{r=0}^j (-1)^r \binom{m+1}{r} \sum_{k=0}^{(j+1-r)b-1} \binom{m-1-k+(j+1-r)b}{m} v_{ik}. \end{aligned}$$

Let  $K = (j+1-r)b - 1$ . The inner sum,  $\sum_{k=0}^K \binom{m+K-k}{m} v_{ik}$ , is the convolution of the sequences (in  $k$ )  $\langle \binom{m+k}{m} \rangle$  and  $\langle v_{ik} \rangle$  evaluated at  $K$ .

We know that

$$\sum_{k \geq 0} \binom{m+k}{m} x^k = (1-x)^{-m-1},$$

and we have the generating function of  $\langle v_{ik} \rangle$  in (9). So the inner sum is equal to the coefficient of  $K$  in

$$(1-x)^{-m-1} \cdot x^{-1} (1-x)^{m+1} \left(x \frac{d}{dx}\right)^{m-i} (1-x)^{-1} = x^{-1} \left(x \frac{d}{dx}\right)^{m-i} (1-x)^{-1},$$

i.e., by (8),

$$\sum_{k=0}^K \binom{m+K-k}{m} v_{ik} = (K+1)^{m-i} = ((j+1-r)b)^{m-i}.$$

Therefore,

$$\begin{aligned} b^{-m} \sum_{r=0}^j (-1)^r \binom{m+1}{r} \sum_{k=0}^K \binom{m+K-k}{m} v_{ik} &= b^{-m} \sum_{r=0}^j (-1)^r \binom{m+1}{r} ((j+1-r)b)^{m-i} \\ &= b^{-i} \sum_{r=0}^j (-1)^r \binom{m+1}{r} (j+1-r)^{m-i} = b^{-i} v_{ij}. \quad \blacksquare \end{aligned}$$

Theorem 3 tells us that the rows of  $\mathbf{V}$  are left eigenvectors of  $\mathbf{\Pi}$  corresponding to the eigenvalues  $1, b^{-1}, \dots, b^{-(m-1)}$ , so Conjectures 1 and 2 are true. Also, as noted above, the formula for  $v_{0j}(m)$  is the same as the explicit formula (5) for the Eulerian numbers, so Conjecture 4 is true. Finally, letting  $i = m - 1$  in (9), we get

$$\sum_{j \geq 0} v_{m-1,j}(m) x^j = x^{-1} (1-x)^{m+1} x \frac{d}{dx} (1-x)^{-1} = (1-x)^{m-1},$$

so Conjecture 3 follows, by the binomial theorem.

There are other patterns in  $\mathbf{V}$ . It is easy to verify that the leftmost column of  $\mathbf{V}$  is all 1's. It is a little harder to show that the rightmost column has  $+1$ 's and  $-1$ 's alternately, but it is a splendid opportunity to use the calculus of finite differences: Let  $E$  denote the advancing operator ( $Ef(x) = f(x+1)$ ),  $I$  the identity operator ( $If(x) = f(x)$ ), and  $\Delta$  the forward difference operator,  $\Delta = E - I$  ( $\Delta f(x) = f(x+1) - f(x)$ ). We will use the binomial operator expansion  $\Delta^n = \sum_{r=0}^n \binom{n}{r} (-I)^r E^{n-r}$  with  $n = m + 1$  and the observation that  $\Delta^{m+1} p(x) = 0$  for any polynomial of degree at most  $m$ . Now, by Theorem 3,

$$\begin{aligned} v_{i,m-1}(m) &= \sum_{r=0}^{m-1} (-1)^r \binom{m+1}{r} (m-1+1-r)^{m-i} \\ &= \sum_{r=0}^{m+1} (-1)^r \binom{m+1}{r} (-1+m+1-r)^{m-i} - 0 - (-1)^{m+1} \binom{m+1}{m+1} (-1)^{m-i} \\ &= \Delta^{m+1} x^{m-i} \Big|_{x=-1} + (-1)^i = (-1)^i. \end{aligned}$$

### THE RIGHT EIGENVECTOR MATRIX $\mathbf{U}$ : EMPIRICAL RESULTS

Let's look at the right eigenvectors of the transition matrix  $\mathbf{\Pi}$ . As an alternative to direct computation of the eigenvectors, we may compute the inverse of the matrix  $\mathbf{V}$ , whose rows are left eigenvectors. Numerical experimentation reveals that, in order to get integer values, we should multiply by  $m!$ , so we let  $\mathbf{U} = m! \mathbf{V}^{-1}$ . For  $m = 2, 3, 4, 5$ , we find that  $\mathbf{U}$  is:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 2 \\ 1 & 0 & -1 \\ 1 & -3 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 6 & 11 & 6 \\ 1 & 2 & -1 & -2 \\ 1 & -2 & -1 & 2 \\ 1 & -6 & 11 & -6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 10 & 35 & 50 & 24 \\ 1 & 5 & 5 & -5 & -6 \\ 1 & 0 & -5 & 0 & 4 \\ 1 & -5 & 5 & 5 & -6 \\ 1 & -10 & 35 & -50 & 24 \end{bmatrix}.$$

Tantalizing patterns are already visible in these first few examples. Even though the columns are the eigenvectors, the top and bottom *rows* leap out at the combinatorial *cognoscenti*: They are Stirling numbers of the first kind! The pattern of the eigenvector in the last column may be exposed by dividing by  $(m-1)!$ : reciprocals of binomial coefficients with alternating signs! Forming difference tables of the columns reveals more patterns: It appears that the  $j^{\text{th}}$  difference of the  $j^{\text{th}}$  column is a constant— $(-1)^j m! / (m-j)!$ —which would make the  $j^{\text{th}}$  column a polynomial of degree  $j$  in the row index  $i$ . To summarize, for the matrix  $\mathbf{U} = m! \mathbf{V}^{-1}$ , we propose:

**Conjecture 5:** Column  $j$  is a degree- $j$  polynomial function of row index  $i$ .

**Conjecture 6:** The final column is proportional to reciprocals of a row of Pascal's triangle with alternating signs.

**Conjecture 7:** The top row consists of unsigned Stirling numbers of the first kind.

**Conjecture 8:** The bottom row consists of signed Stirling numbers of the first kind.

### STIRLING NUMBERS OF THE FIRST KIND

The first few (unsigned) Stirling numbers of the first kind are listed in the following table.

$n$	$n$	$n$	$n$	$n$	$n$	$n$	$n$
0	1	2	3	4	5	6	
0	1						
1	0	1					
2	0	1	1				
3	0	2	3	1			
4	0	6	11	6	1		
5	0	24	50	35	10	1	
6	0	120	274	225	85	15	1

The Stirling number  $\begin{bmatrix} n \\ k \end{bmatrix}$  may be characterized combinatorially as the number of ways  $n$  objects can be arranged into  $k$  cycles. But for

us, the following algebraic characterizations will be more useful: Rising factorial powers may be represented in terms of ordinary powers by means of unsigned Stirling numbers of the first kind:

$$x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1) = \sum_k \left[ \begin{matrix} n \\ k \end{matrix} \right] x^k; \quad (10)$$

falling factorial powers may be represented in terms of ordinary powers by means of signed Stirling numbers the first kind:

$$x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1) = \sum_k (-1)^{n-k} \left[ \begin{matrix} n \\ k \end{matrix} \right] x^k. \quad (11)$$

(See [9, sect. 6.1] [12, pp. 65–68], or [10, ch. 4].) (The notation  $\left[ \begin{matrix} n \\ k \end{matrix} \right]$  for Stirling numbers of the first kind is Karamata's [9, p. 257], [11]. Then  $(-1)^{n-k} \left[ \begin{matrix} n \\ k \end{matrix} \right]$  is the signed Stirling number of the first kind, sometimes denoted  $S_n^{(k)}$  [1, p. 824] or  $s(n, k)$  [19, p. 18].)

### LOOKING FOR $\mathbf{U}$

How can we find an explicit formula for the matrix  $\mathbf{U}$  of right eigenvectors of  $\mathbf{\Pi}$ ? One way would be to solve  $\mathbf{\Pi U} = \mathbf{U D}$ . (This appears to be very difficult.) Our way is to find  $\mathbf{U}$  by solving  $\mathbf{U V} = \mathbf{m! I}$ . This time we'll tell no tall tales. Instead we'll derive the answer by another heavy-duty application of concrete math.

Fix  $m$ . We want to solve  $\sum_k u_{ik} v_{kj} = m! \delta_{ij}$ . By our formula for  $\mathbf{V}$  (Theorem 3),

$$\begin{aligned} \sum_{k=0}^{m-1} u_{ik} v_{kj} &= \sum_{k=0}^{m-1} u_{ik} \sum_{r=0}^j (-1)^r \binom{m+1}{r} (j+1-r)^{m-k} \\ &= \sum_{r=0}^j (-1)^r \binom{m+1}{r} \sum_{k=0}^{m-1} u_{ik} (j+1-r)^{m-k} \\ &= \sum_{r=0}^j (-1)^r \binom{m+1}{r} \sum_{t=1}^m u_{i, m-t} (j+1-r)^t. \end{aligned}$$

Let

$$U_i(x) = \sum_{t \geq 0} u_{i, m-t} x^t = \sum_{t=1}^m u_{i, m-t} x^t,$$

the generating function of the sequence  $a_t := u_{i,m-t}$  (we set  $u_{im} = 0$  for  $0 \leq i < m$ ; then  $U_i(0) = 0$ ). Substituting, we get

$$\sum_{k=0}^{m-1} u_{ik} v_{kj} = \sum_{r=0}^{j+1} (-1)^r \binom{m+1}{r} U_i(j+1-r).$$

Now we multiply by  $y^j$  and sum over  $j$ . Because  $\mathbf{UV} = \mathbf{m!I}$ ,

$$\sum_{j=0}^{m-1} y^j \sum_{k=0}^{m-1} u_{ik} v_{kj} = m! \sum_{j=0}^{m-1} \delta_{ij} y^j = m! y^i$$

for  $0 \leq i < m$ . On the other hand,

$$\begin{aligned} \sum_{j=0}^{m-1} y^j \sum_{k=0}^{m-1} u_{ik} v_{kj} &= \sum_{j \geq 0} y^j \sum_{r=0}^{j+1} (-1)^r \binom{m+1}{r} U_i(j+1-r) \\ &= y^{-1} \sum_{j \geq 0} \sum_{r=0}^{j+1} (-1)^r \binom{m+1}{r} y^r \cdot U_i(j+1-r) y^{j+1-r} \\ &= y^{-1} \left\{ \sum_{n \geq 0} (-1)^n \binom{m+1}{n} y^n \right\} \left\{ \sum_{n \geq 0} U_i(n) y^n \right\} \\ &= y^{-1} (1-y)^{m+1} \sum_{n \geq 0} U_i(n) y^n. \end{aligned}$$

Therefore,

$$\sum_{n \geq 0} U_i(n) y^n = y(1-y)^{-m-1} \cdot m! y^i = m! y^{i+1} \sum_{r \geq 0} \binom{m+r}{m} y^r,$$

whence,

$$U_i(n) = m! \binom{m-1-i+n}{m},$$

i.e.,

$$\sum_{t=1}^m u_{i,m-t} n^t = m! \binom{m-1-i+n}{m}. \quad (12)$$

By (11), we may expand a binomial coefficient in terms of powers:

$$\binom{x}{m} = \frac{x^m}{m!} = \frac{1}{m!} \sum_r (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} x^r.$$

So

$$\begin{aligned} m! \binom{m-1-i+n}{m} &= \sum_r (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} (m-1-i+n)^r \\ &= \sum_{r=0}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} \sum_{t=0}^r \binom{r}{t} n^t (m-1-i)^{r-t}, \end{aligned}$$

whence,

$$m! \binom{m-1-i+n}{m} = \sum_{t=0}^m n^t \sum_{r=t}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} \binom{r}{t} (m-1-i)^{r-t}. \quad (13)$$

(We're pleased to see, at last, the anticipated appearance of the Stirling numbers.) The  $t=0$  term in this sum is 0, because, by (11) again,

$$\sum_{r=0}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} (m-1-i)^r = (m-1-i)^m = 0$$

for  $0 \leq i \leq m-1$ . Now, by (12) and (13),

$$\sum_{t=1}^m n^t u_{i,m-t} = \sum_{t=1}^m n^t \sum_{r=t}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} \binom{r}{t} (m-1-i)^{r-t} \quad \text{for } n = 1, \dots, m.$$

Because the Vandermonde matrix of coefficients  $[n^t]$  ( $1 \leq n, t \leq m$ ) is nonsingular ([12, p. 36]), we conclude that

$$u_{i,m-t} = \sum_{r=t}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} \binom{r}{t} (m-1-i)^{r-t}.$$

This is the formula we sought.

#### THE RIGHT EIGENVECTORS

**Theorem 4.** *Let  $\mathbf{V}$  be the  $m \times m$  matrix  $[v_{ij}]$  where*

$$v_{ij} = v_{ij}(m) = \sum_{r=0}^j (-1)^r \binom{m+1}{r} (j+1-r)^{m-i}$$

for  $0 \leq i, j < m$ . Then  $m! \mathbf{V}^{-1} = [\mathbf{u}_{ij}]$  where

$$u_{ij} = u_{ij}(m) = \sum_{r=m-j}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} \binom{r}{m-j} (m-1-i)^{r-(m-j)}$$

for  $0 \leq i, j < m$  and where  $0^0$  is taken to be 1.

This formula is complicated enough that it still takes some work to verify our conjectures. Conjecture 5 is left as an exercise, and we turn to Conjecture 6: For  $i = 0, \dots, m - 1$ ,

$$\frac{u_{i,m-1}(m)}{m!} = \frac{(-1)^i}{m \binom{m-1}{i}}.$$

To prove this, let  $j = m - 1$  in Theorem 4. We get

$$\begin{aligned} u_{i,m-1}(m) &= \sum_{r=1}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} \binom{r}{m - (m-1)} (m-1-i)^{r-(m-(m-1))} \\ &= \sum_{r=1}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} r (m-1-i)^{r-1} \\ &= \frac{d}{dx} \left( \sum_{r=1}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} x^r \right) \Big|_{x=m-1-i} \\ &= \frac{d}{dx} x^m \Big|_{x=m-1-i} \quad [\text{by (11)}]. \end{aligned}$$

But

$$\frac{d}{dx} x^m = \frac{d}{dx} \prod_{k=0}^{m-1} (x-k) = \sum_{k=0}^{m-1} 1 \cdot \prod_{r \neq k} (x-r),$$

so

$$\frac{d}{dx} x^m \Big|_{x=m-1-i} = (m-1-i)(m-2-i) \cdots (1)(-1)(-2) \cdots (i) = \frac{m!(-1)^i}{m \binom{m-1}{i}}.$$

Conjecture 7 claims that the top row of  $\mathbf{U}$  contains unsigned Stirling numbers of the first kind. It does contain them—in reverse order: For  $j = 0, \dots, m - 1$ ,

$$u_{0j}(m) = \begin{bmatrix} m \\ m-j \end{bmatrix}.$$

It is neat to see how this conjecture reduces to the two basic identities relating Stirling numbers of the first kind to factorial powers. By Theorem 4, for  $n = 1, \dots, m$ ,

$$u_{0,m-n}(m) = \sum_{r=n}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} \binom{r}{n} (m-1)^{r-n}.$$

Thus, this conjecture is equivalent to the identity,

$$\sum_{r=n}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} \binom{r}{n} (m-1)^{r-n} = \begin{bmatrix} m \\ n \end{bmatrix}. \quad (14)$$

Fix  $m \geq n$ , and let

$$a_n = \sum_{r=n}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} \binom{r}{n} (m-1)^{r-n}.$$

The generating function of  $\langle a_n \rangle$  is

$$\begin{aligned} \sum_{n \geq 0} a_n x^n &= \sum_{r=0}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} \sum_{n=0}^r \binom{r}{n} x^n (m-1)^{r-n} \\ &= \sum_{r=0}^m (-1)^{m-r} \begin{bmatrix} m \\ r \end{bmatrix} (x+m-1)^r \\ &= (x+m-1)^m \quad [\text{by (11)}] \\ &= (x+m-1)(x+m-2) \cdots (x) \\ &= x^{\overline{m}}, \end{aligned}$$

which is the generating function of  $\left\langle \begin{bmatrix} m \\ n \end{bmatrix} \right\rangle$ , by (10). Therefore,

$a_n = \begin{bmatrix} m \\ n \end{bmatrix}$ , i.e., identity (14) holds, so Conjecture 7 is true.

The proof of Conjecture 8, which says that the bottom row of  $\mathbf{U}$  contains signed Stirling numbers of the first kind (in reverse order), is much easier, and is left as an exercise.

### COMBINATORIAL IDENTITIES

Many people are fascinated by combinatorial identities, so let's pause to take stock of the relationships associated with  $\mathbf{\Pi}$ ,  $\mathbf{V}$ , and  $\mathbf{U}$ . First let's revisit the Eulerian recurrences (2) and (6). Let

$$\left\langle \begin{matrix} m \\ j \end{matrix} \right\rangle_i = v_{ij}(m) = \sum_{r=0}^j (-1)^r \binom{m+1}{r} (j+1-r)^{m-i}, \quad (15)$$

and notice that the last expression is defined for  $i, m \in \mathbb{R}$  and  $j \in \mathbb{Z}$ . Then

$$\left\langle \begin{matrix} m \\ j \end{matrix} \right\rangle_i = (j+1) \left\langle \begin{matrix} m-1 \\ j \end{matrix} \right\rangle_i + (m-j) \left\langle \begin{matrix} m-1 \\ j-1 \end{matrix} \right\rangle_i. \quad (16)$$

(The proof is left as an exercise.) Thus, for each  $i$ , we have an array of numbers satisfying the Eulerian recurrence relation (2). Furthermore, the recurrence (16) may be used to generate sequences for

$m = 1, 2, 3, \dots$  starting with the values determined by (15) for  $m = 0$ :

$$\left\langle \begin{matrix} 0 \\ j \end{matrix} \right\rangle_i = \begin{cases} 0 & \text{if } j < 0; \\ 1 & \text{if } j = 0; \\ (j+1)^{-i} - j^{-i} & \text{if } j > 0. \end{cases}$$

When  $i = 0$ , one generates the usual Eulerian numbers.

A host of other identities—including familiar ones—arise from the matrix equations  $\mathbf{\Pi U} = \mathbf{U D}$ ,  $\mathbf{V \Pi} = \mathbf{D V}$ ,  $\mathbf{U V} = \mathbf{m! I}$ , and  $\mathbf{V U} = \mathbf{m! I}$ . For example, setting  $i = 0$  and  $j = 0$  in  $\sum_k u_{ik} v_{kj} = m! \delta_{ij}$ , we get the sum formula for unsigned Stirling numbers,  $\sum_k \left[ \begin{matrix} m \\ m-k \end{matrix} \right] = \sum_r \left[ \begin{matrix} m \\ r \end{matrix} \right] = m!$ ; setting  $i = 0$  and  $j = m-1 > 0$ , we get the sum formula for signed Stirling numbers,  $\sum_k (-1)^k \left[ \begin{matrix} m \\ m-k \end{matrix} \right] = 0$ . Similarly, setting  $i = j = 0$  in  $\sum_k v_{ik} u_{kj} = m! \delta_{ij}$ , we get (3)  $\sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle = m!$  again. A more interesting result arises when  $i = 0$  and  $j = m-1 > 0$ :

$$\sum_{k=0}^{m-1} (-1)^k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle / \binom{m-1}{k} = 0.$$

Another identity is inspired by (12). Let

$$w_{kr}(n) = \frac{1}{n!} u_{n-1-k, n-r}(n) = \frac{1}{n!} \sum_{t=r}^n (-1)^{n-t} \left[ \begin{matrix} n \\ t \end{matrix} \right] \binom{t}{r} k^{t-r}.$$

Then one may verify that

$$\binom{x+k}{n} = \sum_{r \geq 0} w_{kr}(n) x^r \quad \text{for } 0 \leq k < n.$$

The Worpitzky identity (4) expressed powers in terms of consecutive binomial coefficients; this identity expresses the binomial coefficients in terms of powers.

### STIRLING NUMBERS OF THE SECOND KIND

So far we've seen binomial coefficients, Eulerian numbers, and Stirling numbers of the first kind. What about Stirling numbers of the second kind? Are they lurking nearby? Recall that the Stirling number of the second kind  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$  (in Karamata [11] notation) counts the

number of ways a set of  $n$  objects can be partitioned into  $k$  disjoint nonempty subsets. They may be calculated by means of the recurrence

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\},$$

which is valid for integer  $n > 0$ , and the special values  $\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = \delta_{n0}$ . An

example we'll use is  $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \binom{n}{2}$ , which is true because there are  $\binom{n}{2}$  ways to choose the one 2-element set. Alternatively, the Stirling numbers of the second kind may be characterized as the coefficients in the representation of ordinary powers in terms of factorial powers:

$$x^n = \sum_k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\underline{k}} = \sum_k (-1)^{n-k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\overline{k}}.$$

Also, they may be represented in closed form as follows:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_r (-1)^{k-r} \binom{k}{r} r^n.$$

More material on Stirling numbers of the second kind may be found in, e.g., [9, sect. 6.1] or [10, chapter 4].

#### MOMENTS OF THE STATIONARY DISTRIBUTION

Now consider the moments of the stationary probability vector. The first moment, the mean, will tell us what the average carry value is, and the second central moment, the variance, will tell us how much the carry values vary about the mean.

**Theorem 5.** *The stationary probabilities*

$$p_j = \frac{v_{0j}(m)}{m!} = \frac{1}{m!} \left\langle \begin{matrix} m \\ j \end{matrix} \right\rangle \quad (j = 0, \dots, m-1)$$

have  $n^{\text{th}}$  factorial moment given by  $\left\{ \begin{matrix} m \\ m-n \end{matrix} \right\} / \binom{m}{n}$  if  $m \geq n$ , mean  $(m-1)/2$ , and variance  $(m+1)/12$ . The standardized probability distribution is asymptotically normal as  $m \rightarrow \infty$ .

*Proof.* For fixed  $m$ , the stationary probabilities have the probability generating function

$$g_m(x) = \sum_k p_k x^k = \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \frac{x^k}{m!}.$$

The moments may be found in terms of the derivatives of the probability generating function. (See, e.g., [9, sect. 8.3] or [8, XI.1].) The  $n^{\text{th}}$  derivative is

$$g_m^{(n)}(x) = \frac{1}{m!} \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle k^n x^{k-n} = \frac{n!}{m!} \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \binom{k}{n} x^{k-n}.$$

We will use the identity (see, e.g., [9, p. 269])

$$\sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \binom{k}{n} = (m-n)! \left\{ \begin{matrix} m \\ m-n \end{matrix} \right\},$$

which is valid for  $m \geq n$ . (This may be proved [9, p.310] by taking the  $(m-n)^{\text{th}}$  difference of the Worpitzky identity (4).) The  $n^{\text{th}}$  factorial moment is

$$\begin{aligned} \sum_k p_k k^n &= \sum_k \frac{1}{m!} \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle k^n = g_m^{(n)}(1) \\ &= \frac{n!}{m!} \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \binom{k}{n} \\ &= \frac{n!}{m!} (m-n)! \left\{ \begin{matrix} m \\ m-n \end{matrix} \right\} \\ &= \frac{\left\{ \begin{matrix} m \\ m-n \end{matrix} \right\}}{\binom{m}{n}} = \frac{\left\{ \begin{matrix} m \\ m-n \end{matrix} \right\}}{\binom{m}{m-n}}. \end{aligned}$$

(This is equivalent to [14, sect. 5.1.3(15)].) In particular, the mean value is

$$g_m'(1) = \frac{\left\{ \begin{matrix} m \\ m-1 \end{matrix} \right\}}{\binom{m}{1}} = \frac{\binom{m}{2}}{\binom{m}{1}} = \frac{\frac{m(m-1)}{2}}{m} = \frac{m-1}{2},$$

and the variance is

$$g_m''(1) + g_m'(1) - g_m'(1)^2 = \frac{\left\{ \begin{matrix} m \\ m-2 \end{matrix} \right\}}{\binom{m}{2}} + \frac{m-1}{2} - \left( \frac{m-1}{2} \right)^2.$$

From the recurrence relation, we calculate

$$\left\{ \begin{matrix} m \\ m-2 \end{matrix} \right\} = \frac{m(m-1)(m-2)(3m-5)}{24}.$$

Substituting this in the previous expression and simplifying, we get that the variance is  $(m + 1)/12$ . (Because the terms involving  $m^2$  cancel, this is smaller than might be expected.) The asymptotic normality is proved in [5, pp. 150–154]. ■

There are still more combinatorial connections. The  $n^{\text{th}}$  factorial moment is exactly  $B_n^{(n-m)}$ , where  $B_n^{(k)}$  is the generalized Bernoulli number of  $k^{\text{th}}$  order and  $n^{\text{th}}$  degree (see [5, chapter 15]). Here  $k = n - m \leq 0$ , and

$$B_n^{(n-m)} = \frac{n!}{m!} \Delta^{m-n} 0^m.$$

Also, there are still more patterns to be discovered and explained in this dazzling array of matrices. May you have fun exploring them.

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