Summer Seminar:
Conceptual Climate Models

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In celebration of
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A word from the organizers

We are very excited for you to join our Summer Seminar on Conceptual Climate Models here at Augsburg College. We have five lectures in this seminar (the odd numbered sessions) and four hands on workshops (the even numbered sessions). This packet contains the main activities for the workshops where the participants may wish to get their hands immersed in some conceptual climate models. Along with this package, a set of MATLAB m-files accompanying the exercises will be available online, through the Mathematics and Climate Research Network website mathclimate.org. In its current state, this packet is not yet perfect; we hope that you will help us with your comments and ideas for improvement.

This packet is organized as follows:

1. Chapter 1: MATLAB GUIDES, where we attempt to gently lead beginner MATLAB users through some of the exercises in this package utilizing this powerful software.

2. Chapter 2: WORKSHEETS, where we list the exercises introducing various conceptual climate models.

Throughout this packet, you will find exercises corresponding to the lectures and hands-on sessions scheduled as the following:

Sess1on 1. Lecture on Global Energy Balance Model
Sess2on 2. Hands on session on Global Energy Balance Model
Sess3on 3. Lecture on Zonal Energy Balance Model
Sess4on 4. Hands on session on Zonal Energy Balance Model
Sess5on 5. Lecture on the Green House Gas Effects
Sess6on 6. Hands on session on the Green House Gas Effects
Sess7on 7. Lecture on the Milankovitch cycle
Sess8on 8. Hands on session on the Milankovitch cycle
Sess9on 9. Lecture on the Snowball Earth

We thank the Mathematical Association of America and the Mathematics and Climate Research Network for their generous support to make this workshop style short course possible.
Chapter 1

MATLAB GUIDES

1.1 Hands On Session 2: Global Energy Balance Models

In this introduction you will learn how to do elementary arithmetic, how to graph functions, and how to do matrix operations.

First, locate and open MATLAB. The standard MATLAB set up has five subwindows: Current Folder, Editor, Command Window, Workspace and Command History.

The Current Folder allows you to navigate and view your saved files in your current directory. The Editor is used for writing and editing longer collections of code or functions. Workspace is a running list of all the global variables you currently have defined. Command History is a self-explanatory log of the commands you recently entered in the Command Window. The Command Window is a workspace for entering short commands or calling functions to run. We’ll mostly be working in the Command Window.

To access the MATLAB help files on a Windows machine you can select “Product Help” under the “Help” option on the top menu bar or you can right click on any function you have entered in your command window and choose the “Help on Selection” option. You can also watch demo videos by clicking on the links at the top of the command window.

1. MATLAB can do everything a scientific calculator can do. For the rest of this worksheet, the code you should enter into the command line will be distinguished by font, as below. Try the following commands.

\[
2+5
\]

\[
(6+19)/5
\]

\[
sin(0)
\]

\[
exp(2)
\]

Does MATLAB work in degrees or radians? We can also create and save variables for later use.

This section was prepared by Anna Barry and Samantha Oestreicher
e=exp(1); e^2

Notice that adding a semicolon after a command suppresses the output from the command window. The variable 'e' is now listed in the Workspace subwindow with a value of 2.7183. We can adjust how many digits of the number we see in the command window.

format long
e^2
format short

2. MATLAB is also a graphing calculator. To properly graph in MATLAB we need to create a vector of inputs. There are two ways to create our input vector.

x1=1:0.5:6
x2=linspace(1,6,11)

What is the difference in these two vector creation methods? Next we will plot a cosine function.

figure(1)
plot(x1, cos(x1))

This is not a very smooth graph. We can smooth it out by adding more elements to our input vector.

x3=linspace(1,5,25);
plot(x1, cos(x1), 'c')
hold on
plot(x3, sin(x3), 'r-')
hold off

Notice that the hold command will keep old plots showing on the figure, allowing you to plot more than one data set in a single figure. Depending on your version of Matlab the dash after the r in 'r-' may not work. Check your "plot" help file for other ways to make your plotted line dashed.
3. MATLAB is short for “Matrix Laboratory.” MATLAB was designed for matrix manipulation. MATLAB can multiply matrices traditionally or component wise.

\[
A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}
\]

\[
2 \times A
\]

\[
4 \times A + A
\]

\[
A^2
\]

\[
A.^2
\]

Can you think of any times when component wise multiplication would be valuable? We can also use old matrices in our definition of a new matrix to create a square matrix.

\[
B = \begin{bmatrix} A \\ 8 & 9 & 10 \end{bmatrix}
\]

Now that we have made the matrix a square matrix, we can complete standard matrix multiplication.

\[
B^2
\]

\[
A \times B
\]

\[
\text{ones}(2)
\]

\[
\text{zeros}(2,1)
\]

To clean up our workspace for the next exercises, we can clear our variables one at a time or all at once.

\[
\text{clear A}
\]

\[
\text{clear all}
\]

We are now ready to begin our investigation into global energy balance models. In the next three sections we will consider different formulations of a global energy balance model through the consideration of Earth, Venus and Mars.

**Global Energy Balance Models with Stefan Boltzmann**

First we consider the following the energy balance equation:

\[
R \frac{dT}{dt} = Q(1 - \alpha) - \sigma T^4
\]

(1.1)

where \( T = T(t) \) is the planet’s average surface temperature in kelvins.
1. By hand, show that equation (1.1) admits one positive equilibrium solution \( T^* \). Determine the role played by \( Q \) and \( \sigma \) in determining the magnitude of \( T^* \).

2. Find the linear approximation for equation (1.1) at \( T^* \) by hand. Next we will plot the approximation in MATLAB. Use this plot to help determine whether \( T^* \) is a sink or a source.

To plot the linearization of \( \frac{dT}{dt} \) in MATLAB we need to explicitly define the parameters. Recall \( Q = 343 \frac{W}{m^2} \) and \( \alpha = 0.3 \) for the planet Earth. Stefan Boltzmann’s constant is \( \sigma = 5.67 \times 10^{-8} \frac{W}{m^2K^4} \). \( R \) is heat capacity of the earth. Since this term is difficult to quantify and will not change the qualitative analysis, we assume \( R = 1 \).

For this and future exercises you may find it easier to work in the Editor. Open a new script (by clicking on the white page icon in the MATLAB toolbar) and enter the following lines of code. Save the script to your current directory. Then run the code by Evaluating the cell or clicking the big green arrow.

```matlab
%% This cell contains code to plot dT/dt versus T.
Q=342; %Values of the parameters for Earth.
a = 0.3; %Values of the parameters for Earth.
s=5.67*10^(-8); %Stefan Boltzmann’s constant.
Teq=nthroot(Q*(1-a)/s,4);
slope=-4*s*Teq^3;
T=linspace(Teq-2,Teq+2,30);
plot(T,slope*(T-Teq))
hold on
plot(T, zeros(length(T))) %plots the dT/dt=0 axis.
ylabel('dT/dt')
xlabel('T')

%prints a message to the command line.
strcat('The equilibrium value of T is ', num2str(Teq), ...
' '. The plot is in the figure window')

hold off

%% This begins my next cell.

Notice the above code makes use of the cell feature in MATLAB. Cells can be very useful within a MATLAB script. On a basic level, cells allow you to organize your code into small pieces. More than that, cells allow you to run one section of your code without re-running the whole script. While this may not be necessary in this lab’s exercises, you may find it useful in the future. The ellipse, "...", symbol allows us to break up a long command into two lines.

3. Compare the temperature at equilibrium (Teq) to Earth’s current average temperature of 287.7 K. How might you explain the discrepancy between these two values?
4. Consider the following planetary data for Venus and Mars.

<table>
<thead>
<tr>
<th></th>
<th>$Q \left( \frac{W}{m^2} \right)$</th>
<th>$\alpha$</th>
<th>Avg. surface T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Venus</td>
<td>665</td>
<td>0.71</td>
<td>737K</td>
</tr>
<tr>
<td>Mars</td>
<td>149</td>
<td>0.17</td>
<td>213K</td>
</tr>
</tbody>
</table>

Compute the value of the equilibrium temperature $T^*$ for both Venus and Mars by editing the script you made for Earth. Compare and contrast the computed values with the given average surface temperatures. How might you explain any similarities and/or differences?

**Budyko’s Global Energy Balance Model**

Consider Budyko’s formulation of the 0–D energy balance model.

$$\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$  \hspace{1cm} (1.2)

1. Budyko’s equation is in terms of degrees Celsius. Recall the conversion from Kelvin to Celsius is

$$^\circ C = K - 273.15$$

The values of $A$ and $B$, which measure outgoing longwave radiation, can be observed directly from satellite data and are $A = 202 \frac{W}{m^2}$ and $B = 1.9 \frac{W}{m^2}$. Use this information to solve for the value of $T^*$. Is this closer to the actual value than the 0-D Stephan-Boltzman model was? Why do you think that is?

2. Next we are going to consider several other planets’ equilibrium temperatures. It will help to first create a function which solves for $T^*$ given $Q, \alpha, A$, and $B$. To do this we open a new script in the Editor window and enter the following code.

```matlab
function [Teq]=Tstar(Q,a,A,B)
    Teq=(Q.*(1-a)-A)./B;
end
```

Then save the file as Tstar.m to your current directory. Now in the command window we can compute $T^*$ for Earth by calling the function with specific inputs.

```
Tstar(342,0.3,202,1.9);
```

We can consider multiple inputs by inputting vectors into the function equation.

```
Tstar([342 350],0.3,202,1.9);
```

Consider how $T^*$ will change with a range of planetary albedo values. Recall $\alpha \in [0,1]$, where $\alpha = 0$ is a 100% absorbing black body and $\alpha = 1$ is a completely reflective body. Create a plot in MATLAB that compares the value of $\alpha$ to the value of $T^*$. 


3. Compute $T^*$ for Mars and Venus with the given values of $A$ and $B$, and with the values of $Q$ and $\alpha$ given in the table below. How well do these computations match the surface $T$ on Mars and Venus? Note the values for $A$ and $B$ are tuned for Earth’s atmosphere. Can you think of a way, using MATLAB, to find a more appropriate value of $A$ and $B$ for these planets? If we assume that $B = 1.9 \frac{W}{m^2}$ is constant for all planets, what values of $A$ are appropriate on each planet? What does this assumption on the linear approximation mean about the physical differences in planetary atmospheres? What if $A$ was held constant and $B$ varied between planets? Is there a linear relationship between $A$ and $B$ which must hold for each planet?

<table>
<thead>
<tr>
<th>Planet</th>
<th>$Q$</th>
<th>$\alpha$</th>
<th>surface $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth:</td>
<td>343 W/M²</td>
<td>0.3</td>
<td>14°C</td>
</tr>
<tr>
<td>Mars:</td>
<td>149 W/M²</td>
<td>0.25</td>
<td>-36°C</td>
</tr>
<tr>
<td>Venus:</td>
<td>665 W/M²</td>
<td>0.9</td>
<td>462°C</td>
</tr>
</tbody>
</table>

Global Energy Balance Models and the Goldilocks Zone

“This one is too hot. This one is too cold.” Earth exists at a special place in our solar system. The amount of light received from the sun is just right to have liquid water on our planet—one of the key ingredients to life as we know it. This region of space is called the Goldilocks Zone. If we assume the sun is a perfect black body, then its radiation is modeled by Stefan-Boltzmann’s law.

$$B_S = \sigma T^4$$

where $\sigma = 5.67 \times 10^{-8} \frac{W}{m^2 K^4}$ and the temperature of the sun is $5780 K$. This radiation is spread out over the surface of the sun. The radius of the sun is $r_S = 6.955 \times 10^8 m$.

Total emission of the sun = $4\pi r_S^2 B_S$

These emissions become less intense as the light moves away from the Sun. By the time it reaches Earth, the radiation has spread out evenly over a sphere of radius $r_{ES} = 149.6 \times 10^9 m$, the average distance between the Earth and the Sun. The intensity of solar irradiance at Earth distance is

$$\frac{4\pi r_S^2 B_S}{4\pi r_{ES}^2} = \frac{r_S^2 B_S}{r_{ES}^2}$$

Of this energy, the Earth collects light over an area of the disk of Earth’s radius ($r_E = 6.378 \times 10^6 m$) and spreads it over an area of the surface area of Earth’s spherical surface. Thus,

$$\text{incoming solar radiation} = \frac{4\pi r_S^2 B_S}{4\pi r_{ES}^2} \frac{\pi r_E^2}{4\pi r_E^2} = \frac{r_S^2 B_S}{r_{ES}^2} \frac{1}{4}$$

1. Compute the value of Earth’s incoming solar radiation using the above formula. How close is this to the best estimate of $343 W/m^2$?

2. We assume that liquid water requires the global average temperature to be above the freezing level of sea water $-10^6 C$ and below the boiling point of sea water $101^6 C$. Using the Budyko model, we can determine the equilibrium temperature of the planet
given a specific incoming solar radiation value. Assuming the constants $\alpha$, $A$ and $B$ for Earth in the Budyko model are acceptable for all ranges of incoming solar radiation: How much could $r_{ES}$ change and still allow for liquid water on the planet?

3. Sirius is the brightest star in the night sky with a radius 1.711 times bigger than that of our sun. It has a temperature of 9,940K. For this question we will use Budyko’s model of Earth with the constants $\alpha$, $A$ and $B$ as above.

If Earth was orbiting Sirius, what range of distances would be acceptable for liquid water to exist? What changes if we assume our potential habitable planet is half the size of Earth?
1.2 Hands On Session 4: Zonal Energy Balance Models

Introduction

The following is the zonal (one-dimensional) Budyko energy balance model:

\[
\frac{\partial T}{\partial t} = Q_s(y)(1 - \alpha(y, \eta)) - (A + BT) + C(T - T_0) \tag{1.3}
\]

We recall that \(y\) is the sine of the latitude, \(T(y, t)\) the temperature profile, \(Q_s(y)\) the distribution of the annual mean insolation across latitudes, the function \(\alpha(y, \eta)\) is the albedo at \(y\), when the ice line is at \(\eta\), \(A\) and \(B\) constants in the outgoing longwave radiation (OLR) term, while the final term represents the linear heat transport given by a tendency of the planetary temperature to relax to its global average,

\[
T(t) = \int_0^1 T(y, t)dy \tag{1.4}
\]

For simplicity, we assume the planet is symmetric across the equator and so we only need consider the northern hemisphere, hence the limits of integration above.

Steady States of Budyko’s EBM

1. Computing integrals in MATLAB

Recall that \(s(y) = 1 - \frac{482}{2} (3y^2 - 1)\). MATLAB can do both symbolic and numerical integration.

(a) Verify that \(\int_0^1 s(y)dy = 1\) using both symbolic and numerical integration.

The \texttt{int} command will do both forms of integration:

\begin{verbatim}
syms y
int(1-.482/2*(3*y^2-1))
int(1-.482/2*(3*y.^2-1),y,0,1)
quad (@(y) 1-.482/2*(3*y.^2-1),0,1) (for version 2012b w/o syms)
\end{verbatim}

(b) Suppose we consider an Earth-like planet consisting only of water and ice. Given a single location of the ice line \(\eta \in [0, 1]\), set

\[
\alpha(y, \eta) = \begin{cases} 
\alpha_w, & y < \eta \\
\frac{\alpha_w + \alpha_i}{2}, & y = \eta \\
\alpha_i, & y > \eta 
\end{cases}
\]

Define \(\overline{\alpha}(\eta) = \int_0^1 \alpha(y)s(y)dy\). Suppose the albedo of water \(\alpha_w = 0.32\) and the albedo of ice \(\alpha_i = 0.62\). Compute \(\overline{\alpha}(\eta)\) by hand or in MATLAB. If you would like to do this in MATLAB, you may find the \texttt{inline} command useful; for more information type

This section was prepared by Anna Barry and Samantha Oestreicher
help inline

in the command line.

(c) Using your answer from part (b), calculate $\alpha(0.2)$, $\alpha(0.6)$, and $\alpha(0.95)$ and interpret your results.

2. Suppose $T^* = T^*(y, \eta)$ denotes the steady state of equation (1.3) given that the ice line is at $\eta$ In other words, for a fixed $\eta$, $T^*(y, \eta)$ solves $\frac{\partial T^*}{\partial y} = 0$.

(a) By hand, find an expression for the equilibrium global average temperature, $\overline{T^*}(\eta)$, as a function of $\overline{\alpha}(\eta)$. Hint: Plug in the equilibrium solution in (1.3) and integrate both sides of the equation from equator to pole.

(b) Set $Q = 343$, $A = 202$, $B = 1.9$, $\alpha_w = 0.32$ and $\alpha_i = 0.62$. Calculate and interpret $\overline{T^*}(\eta)$ for $\eta = 0.2$, 0.6, and 0.95.

3. (a) Use your expression for $\overline{T^*}(\eta)$ from Exercise 2 to solve for the equilibrium temperature profiles $T^*(y, \eta)$.

(b) In MATLAB, plot $T^*(y, \eta)$ for several values of the ice line $\eta$. As you have seen in the first hands-on session, there is more than one way to plot a function. In general, you will need to discretize the independent variable; here is some sample code, executable as a Matlab function.

```matlab
%%Budyko_Equilibrium_Plots.m
% This cell will create equilibrium temperature profile plots
% for the one-dimensional Budyko-Sellers model with linear heat transport
% Model Parameters
Q = 343;%Solar constant in Watts per m^2
s0 = 1;%Insolation coefficient 1
s2 = -0.482;%Insolation coefficient 2
aw = 0.32;%Albedo of open ocean
ai = 0.62;%Albedo of sea ice
A = 202;%Constant related to OLR in Watts per m^2
B = 1.9;%Constant related to OLR in Watts per m^2 per degC
C = 3.04;%Constant for heat transport in Watts per m^2 per degC
eta = 0.95;%Location of the ice line
%Discretize y
yw=0:0.01:eta;
iy=eta:0.01:1;
%Insolation distribution s(y)
sw = s0+s2/2*(3*yw.*yw-1);
si = s0+s2/2*(3*yi.*yi-1);
%alphabar at eta
alphabar = aw*(-(eta*(241*eta^2 - 1241))/1000)+...
ai*((eta*(241*eta^2 - 1241))/1000 + 1);
%Equilibrium Global Average Temperature Tbarstar
Tbarstar = 1/B*(Q*(1-alphabar)-A);
%Equilibrium Profile Tstar
```
\[ T_{\text{star1}} = \frac{Q}{B+C} \left( sw(1-aw)+C/B(1-\text{alphabar}) \right) - A/B; \]
\[ T_{\text{star2}} = \frac{Q}{B+C} \left( si(1-ai)+C/B(1-\text{alphabar}) \right) - A/B; \]

plot(yw,T_{\text{star1}});
hold on
plot(yi,T_{\text{star2}});
hold off

(c) It can be shown that if the ice line \( \eta \) is fixed, then

\[ T(y,t) \to T^*(y,\eta) \text{ as } t \to \infty. \]

Compute by hand the ice line temperature of the steady state \( T^*(\eta,\eta) \). Note that this will be a real valued function of \( \eta \) defined on the unit interval.

So far we have seen there is an equilibrium temperature profile \( T^*(y,\eta) \) for each value of the ice line \( \eta \). However, we have not allowed for the possibility that ice forms or melts, nor have we identified which of the ice lines are important for the dynamics of the system. For example, today the ice line satisfies \( \eta \approx 0.95 \) and so one might expect this to be a stable steady state. We address each of these issues in the following sections.

**Modeling the ice line dynamics**

We introduce an equation which allows ice line to advance (toward the equator) and retreat (toward the pole) by making the following assumption from [3]:

- If the temperature at the ice line is higher than the critical temperature, then ice line retreats. If it is below the critical temperature, ice forms.
- Ice line moves at a much slower speed relative to the temperature profile.

4. Suppose that \( T_c \) denotes the critical temperature and \( \epsilon \) denotes the time constant for \( \eta \).

(a) Write an equation for \( \frac{d\eta}{dt} \)

(b) Recall that the temperature profile \( T(y,t) \to T^*(y,\eta) \) as \( t \to \infty \). While this question will be explored further in the next question, you might want to first discuss the dynamics of \( \eta \) as the temperature profile reaches a steady state \( T^*(y,\eta) \) at this moment. Hint: As \( t \to \infty \), \( T(\eta,t) \to T^*(\eta,\eta) \).

**Stability**

In the previous exercise, you might arrive at an equation for the ice line \( \eta \) similar to the following:

\[
\frac{d\eta}{dt} = \epsilon(T(\eta,t) - T_c)
\]  
(1.5)
5. Explain the equation (1.5)

When equation (1.5) for the ice line is coupled with equation (1.3) for the temperature profile, the resulting system may be written in the fast time scale $t$ as:

$$\frac{\partial T}{\partial t} = Qs(y)(1 - \alpha(y, \eta)) - (A + BT) + C(T - T) \tag{1.6}$$

$$\frac{d\eta}{dt} = \varepsilon(T(\eta, t) - T_c) \tag{1.7}$$

with $\varepsilon \ll 1$

By rescaling time to $\tau = \varepsilon t$, the system (1.6) may be rewritten in the slow time scale $\tau$ as:

$$\varepsilon \frac{\partial T}{\partial \tau} = Qs(y)(1 - \alpha(y, \eta)) - (A + BT) + C(T - T) \tag{1.8}$$

$$\frac{d\eta}{d\tau} = (T(\eta, \tau) - T_c) \tag{1.9}$$

and again, $\varepsilon \ll 1$.

When $\varepsilon = 0$, the solution to the steady state $\frac{\partial T}{\partial \tau} = 0$ is $T^*(y, \eta)$. Therefore, the system (1.8) reduces to

$$\dot{\eta} = T^*(\eta, \eta) - T_c. \tag{1.10}$$

6. Consider the reduced equation (1.10).

(a) Using your previous computation of $T^*(\eta, \eta)$ to compute the equilibria for $\eta$. Hint: There are two in the unit interval, based on the reduced equation (1.10), when $\varepsilon = 0$.

Notes: In the case when $\varepsilon > 0$ but is very small, the ice line dynamics is essentially similar to the $\varepsilon = 0$ case. For more details, see reference [9].

(b) Decide the stability of each equilibrium you found in part a. Hint: draw the phase line of $\eta$. You may find the code below helpful.

```matlab
%%Teta.m
% This cell will plot values of
% the temperature equilibrium profile at the ice line
% and the line Tc=-10 for the one-dimensional Budyko-Sellers model
% with linear heat transport.
% Model Parameters
Q = 343;%Solar constant in Watts per m^2
s0 = 1;%Insolation coefficient 1
s2 = -0.482;%Insolation coefficient 2
aw = 0.32;%Albedo of open ocean
ai = 0.62;%Albedo of sea ice
A = 202;%Constant related to OLR in Watts per m^2
B = 1.9;%Constant related to OLR in Watts per m^2 per degC
```
C = 3.04; % Constant for heat transport in Watts per m^2 per degC  
eta = 0.95; % Location of the ice line  
% Critical temperature  
Tc = -10;  
% Discretize eta  
eta = 0:0.001:1;  
% Insolation at the ice line eta  
seta = s0 + s2/2*(3*eta.*eta - 1);  
% alphabar at eta  
alphabar = aw*(-(eta.*(241*eta.^2 - 1241))/1000)...  
+ ai*((eta.*(241*eta.^2 - 1241))/1000 + 1);  
% Equilibrium Profile Tstar as a function of ice line  
teta = (Q/(B+C)*(seta.*(1+(-aw-ai)/2)+C/B*(1-alphabar))-A/B);  
plot(eta,teta);  
hold on  
plot(eta,Tc);  
hold off

Green House Gas Effects

7. Exploring greenhouse gas effects on equilibria. In Exercise 5, you found two physically reasonable equilibria. Recall that the Budyko model has an OLR term \((A + BT)\). In particular, notice that an increase in the parameter \(A\) amounts to an increase in heat re-radiated back to space, and hence \(T\) decreases. It is common to relate the parameter \(A\) to the amount of greenhouse gases in the atmosphere, especially carbon dioxide. For more on this, see for example [1], [2].

How does increasing and decreasing \(A\) affect the equilibria you found in Exercise 4? Identify any approximate values of \(A\) that give bifurcations and interpret the meaning of these.
1.3 Hands On Session 6: The Greenhouse Effect

MATLAB Ordinary Differential Equations (ODE) Solver

MATLAB has many built in ODE solvers. If you go to product help and type in ode solver, you will find an entry solver, which has information of all the different ODE solvers in MATLAB. In this exercise we will use one particular solver, ode45.

To use the solver you simply need two m-files: the main file that calls the ode solver, and the system file that houses the ODE system. As an example, you may find the files bud_main_45.m and bud_sys.m in the MATLAB m-files folder, properly named to remind us what each file does.

Let’s walk through how these files are created. First write down the system of differential equations that we want to solve numerically. In the case of the two files above, you may check the system file and discover that the system simulates exercise 2.15 as in the exercises in section 2.2, in yearly increments:

\[
\dot{w} = \frac{K}{R}Q(1 - \alpha_0) - A - (B + C)w + C \mathcal{T}(\eta) \tag{1.11}
\]

\[
\dot{\eta} = K \epsilon(T(\eta) - T_c) \tag{1.12}
\]

Once we have this ODE system written out, we create an m-file for the system as a function with the output being \(\dot{w}\) and \(\dot{\eta}\).

```matlab
function Xdot= bud_sys(t,X)
%% MAA short course 2013
% Created by Esther Widiasih
%% Setting Parameters
epsilon=3.9e-13;
A = 202; \%Watts/m^2
B = 1.9; \%Watts/m^2 C
C = 3.04; \%Watts/m^2 C
Q=343; \%Watts/m^2
s_2=-0.482;
R = 4e10; \%Joules/m^2C
K = 3.16e7; \%seconds
Tc = -10;
alpha1 = .32;
alpha2 = .62;
alpha0 = (alpha1+alpha2)/2;
%% The variables
```

This section was prepared by Esther Widiasih
eta = X(1);
w = X(2);
p2=(1/2)*(3*eta.^2-1);
P2=(1/2)*(eta.^3-eta);
Teta=w+Q*s_2*(1-alpha0)*p2/(B+C);
Tbar=w+(eta-(1/2)+s_2*P2)*Q*(alpha2-alpha1)/(B+C);

%%% the etadot and wdot for the current time 't'
%%% from the system of equations
wdot=(K/R)*(Q*(1-alpha0)-A-(B+C)*w+C*Tbar);
etadot=K*epsilon*(Teta-Tc);

%%% the output
Xdot=[etadot; wdot];
end

Once the system file is created, we create the main file that calls MATLAB’s ode45
solver and executes the ODE system. Note we must first decide on the initial conditions
and the desired time span.

%%% Experiment time set up
starttime=1;
endtime=1e6;
%%% Set the initial conditions to run using ODE45
%
x0 = [0.7; 10];
t0=starttime;
tf=endtime-1;

The syntax to run the ode45 solver is as follows:

%%% Run ode45
[T F] = ode45(@(t,W) bud_sys(t,W),[t0,tf],x0);

The outputs are the time array T and the phase space array F. In this example F has
two columns, the first column for w and the second for η. Having these outputs, we can
now plot the η(t), T(t) and the “phase plane” T − η.

%%% using the result to plot Tbar, eta etc.
% May need some parameters, be sure to use the same one as in sys
B = 1.9; %Watts/m^2
C = 3.04; %Watts/m^2
Q=343;
s_2=-0.482;
alpha1 = .32;
alpha2 = .62;
w=F(:,2);
etas=F(:,1);
P2=(1/2)*(eta.^3-eta);
Tbar=w+(eta-(1/2)+s_2*P2).*Q*(alpha2-alpha1)/(B+C);

%%% plotting Tbar, eta over the time
figure
plot(T,F(:,1))
ylabel('eta')
xlabel('time')
title('The ice line evolution over time')
figure
plot(T,Tbar);
ylabel('Tbar')
xlabel('time')
title('The global mean temperature over time')

figure
plot(eta,Tbar);
ylabel('eta')
xlabel('Tbar')

Exercises

1. Warm up. Change the initial conditions so that \( \eta(0) = \eta(0) = 1, 0.5, 0.1 \) and the non-physical conditions \( \eta(0) = 1.2, -0.2 \). What do you notice?

2. Jog. Change the parameter \( A \) to values close to some values in the range \( 140 < A < 220 \). What do you notice?

3. Sprint. Recall that \( A \) may be thought to represent a greenhouse gas parameter. Suppose that

\[
\dot{A} = \delta(\eta - \eta_c)
\]

based on Kirschvink’s idea as in Exercise 15 of Section 2.7. Here, our task is to modify the m-files above to numerically solve the \( \eta - w - A \) system. Use \( \delta = 1e - 3 \cdot \epsilon \) and \( \eta_c = 0.8 \), then again with \( \eta_c = 0.67 \), and the initial conditions \( A(0) = 202, \eta(0) = 0.9 \) and \( w(0) = 0 \).
1.4 Hands On Session 8: The Milankovitch Cycles

We begin our investigation of the Milankovitch cycles by working with the Milankovitch cycle data as computed by Laskar. Earth’s obliquity and eccentricity data may be downloaded from Laskar’s webpage: http://www.imcce.fr/Equipes/ASD/insola/earth/La2004/index.html. We will be working from the data.mat file provided with this lab.

Place the data.mat file in the MATLAB directory you are currently working in. To load the data into MATLAB use the following command:

```matlab
load data.mat
```

Open the data file in the Variable Editor by double clicking the data variable in the Workspace or by highlighting the data variable in the Workspace and clicking the Open Selection button at the top of the Workspace mini-window. Notice the data consists of four columns. The first column is time in kyr. The computations span from 10,000,000 BCE to 5,000,000 CE. The second column contains eccentricity data. The third column contains obliquity. The fourth column contains the sea sediment data. ‘NaN’ indicates no data for that time.

1. As a first attempt, we can use the Variable Editor to select a collection of rows which are associated to a particular period of time to plot. If we wanted to plot from 8 million years ago to 6 million years ago, then we find the row value which has −8000 and −6000 in the first column respectively. For these values we would want the range of rows from 2001:4001. What rows would be appropriate if we wanted to consider the data from 2 mya to present?

2. Plotting the data. We begin by plotting the second column of data from 2 million to 1 million years ago.

```matlab
figure(1)
plot(data(8001:9001,2))
```

Plot the third column of data for the time period from present day to 1 million years from now.

3. Use the subplot function to plot the eccentricity (2nd) and obliquity (3rd) columns on separate axes for the time period from 2,000,000 to present. The help menu may be used to find syntax information for the subplot function. Which forcing has a higher frequency?

4. The fourth column contains the δ^{18}O data from the Lisiecki-Raymo stack interpolated to 1 kyr increments. Determine how far back in Earth’s history the stack extends and plot that data. Because δ^{18}O gives a measure of how much ice there is, δ^{18}O goes down when the temperature increases. Thus it useful to consider the negative of the δ^{18}O. Try plotting eccentricity or obliquity along with −δ^{18}O data. Is there any relationship between eccentricity or obliquity and the −δ^{18}O data?

---

This section was prepared by Richard McGehee and Samantha Oestreicher
5. Using the graph and any other methods you can think of answer the following questions for eccentricity and obliquity.

(a) What is the approximate period the forcing?
(b) Is Earth near a maximum or minimum value of the forcing?
(c) What does this value of the forcing mean for the planet physically?
(d) Are there any other larger features of the forcing besides the “fast” oscillations?
(e) Are there any obvious correlations or connections between eccentricity and obliquity?

PaleoBudyko

For this section we will consider the interaction between the Milankovitch cycles and Earth’s climate. We will be using a GUI created by Richard McGehee provided with this lab. Be sure the file PaleoBudyko.m is in your current directory along with the data.mat file we worked with in the previous section.

To begin the Budyko user interface use the following command in the Command Window.

PaleoBudyko

This should open a new window (Figure 1) with three graphs stacked vertically with blue and green plots. This is an interactive interface to help us with our investigation. To avoid error messages, use the 'Exit' button to close the window and stop the program.

The top graph shows the plots of eccentricity and obliquity used for this computation. The second graph shows the output of the Budyko model; the change in ice line value is plotted in green while global mean temperature (GMT) is plotted in blue. Lastly, the bottom graph shows the \( \delta^{18}O \) data for the selected time range when applicable.

1. The interface is currently set at default values. Each graph is showing the time series of the data from 5 million years ago to present with Earth’s values of \( A, B \) (for outgoing long-wave radiation), \( C \) (for the power of the transport), \( T_c \) (ice line temperature), \( \alpha_1 \) and \( \alpha_2 \) (the albedo above and below the ice line). At any point the value of a parameter can be returned to the default value by clicking the adjacent Default button.

To begin, we will adjust the time range to be from \(-2000\) to \(0\). This will cover the time from 2 million years ago to present. Change the appropriate boxes and click the green Refresh button to see the change. Do the eccentricity and obliquity values in the top graph look similar to what you plotted earlier?

2. Is Budyko more closely following eccentricity or obliquity? This is a difficult question to answer. One way to consider this problem is by considering the frequency signal of the forcing and model output. We do this by using the dropdown menus on the upper left of each graph to select the Power Spectrum for each graph. Click the Refresh button to see the power spectrum. Notice that Budyko has a power spectrum which looks very similar to that of obliquity. Why should we expect this? How does this vary from the power spectrum of the \( \delta^{18}O \) data?
3. Consider the plots for the late Pliocene (3−2.5 mya), early Pleistocene (2.5−1.2 mya), and late Pleistocene (1.2−present mya). How does the time series of the Budyko model and $\delta^{18}$O change between these time periods? What is the dominant frequency of the early Pleistocene? How does this frequency change in the late Pleistocene? Use these observations to formulate an argument for why a model which only contains ice-albedo feedback is not enough to explain Earth’s climate.

4. **Ice free world.** Consider a time period in our world where $\alpha_1 = \alpha_2$. This change in the model has a physical interpretation that ice never grows. This was the case for our world before 35 million years ago. Using eccentricity and obliquity values from an arbitrary point of time: How does an ice free world respond to eccentricity versus obliquity? How and why is this different from our current ice age climate?

5. **Changing Carbon.** We know that $A$ and $B$ control the outgoing long-wave radiation and that $A$, specifically, has a connection to atmospheric CO$_2$. How does changing $A$ effect the Budyko model output?

6. Explore a parameter or question of your choice. Are there any questions from lecture which you can explore using this interface? Some possible questions include:

   (a) How does the relative power of the transport term ($C$) change the results?
   (b) What would happen if the ice line was created at a different temperature ($T_c$)? What would this mean physically and how would the Budyko model interpret this?
   (c) Imagine an extrasolar planet with a surface and atmosphere different from Earth’s, but with a star similar to the Sun and with an orbit similar to Earth’s. How would the climate respond to Milankovitch cycles?
   (d) What does the Budyko model predict for the future of our planet Earth? In your opinion, how likely is this predicted scenario?
Chapter 2

Worksheets

This chapter contains exercises, some of which were treated in the MATLAB GUIDE chapter. All of the exercises, however, could be done using software other than MATLAB. The session number most pertinent to each exercise is indicated, though participants are encouraged to be adventurous and try them out at any time (albeit, at their own peril).

1. This exercise complements those in Budyko’s 0-D Energy Balance Model in Session 2. Recall that the $\sigma T^4$-term in equation (1.1) models the outgoing longwave radiation (OLR) emitted by the planet. Suppose we model the OLR via a linear term of the form $A + BT$, with $B$ a positive constant, thereby arriving at the energy balance equation

$$R \frac{dT}{dt} = Q(1 - \alpha) - (A + BT).$$

The temperature $T$ is now given in °C.

(a) Explain, in terms of the model, the requirement that $B > 0$.

(b) Find the general solution of equation (2.1). What is the behavior of solutions over time?

(c) The parameters $A$ and $B$ have been estimated via satellite measurements as a function of the surface temperature for the planet Earth. The values are $A = 202$ W m$^{-2}$ and $B = 1.9$ W m$^{-2}$ (°C)$^{-1}$.

(i) Using $Q = 342$ W m$^{-2}$ and $\alpha = 0.3$, compute the Earth’s average surface temperature at equilibrium $T^*$ per equation (2.1). Why might you expect this value to be fairly close to 15°C, the Earth’s current average temperature?

(ii) How does the magnitude of $T^*$ vary with the parameters $A$ and $B$? Discuss in the context of the OLR term in the model.

(iii) Assume the albedo of open water is $\alpha = \alpha_w = 0.32$. Compute $T^*$ in the case where the Earth is ice free, that is, with $\alpha$ in equation (2.1) replaced by $\alpha_w$.

The exercises in this chapter were prepared by James Walsh and Esther Widiasih, with much help from the rest of the team.
With the temperature governed by equation (2.1), would ice ever form if the Earth were to become ice free? (Assume ice forms when \( T < T_c = -10^\circ \text{C} \)).

(iv) Assume the albedo of ice is \( \alpha = \alpha_i = 0.62 \). Compute \( T^* \) in the snowball Earth state, that is, with \( \alpha \) in equation (2.1) replaced by \( \alpha_i \), indicating the planet is completely ice covered. With the temperature governed by equation (2.1), would ice ever melt if the Earth were in a snowball state?

(v) The parameters \( A \) and \( B \) have not been estimated via observations for Venus. Nonetheless, using the values of \( Q \) and \( \alpha \) for Venus given below, determine a relationship between \( A \) and \( B \) so that the temperature at equilibrium on Venus per equation (2.1) is the current average, \( T^* = 464^\circ \text{C} \). If \( B \) were to equal 1.9, find the value of \( A \) so that \( T^* = 464^\circ \text{C} \). In terms of the model, how might you interpret the fact that \( A < 0 \) in this case?

<table>
<thead>
<tr>
<th></th>
<th>( Q ) (W m(^{-2}))</th>
<th>( \alpha )</th>
<th>Ave. surface temp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Venus</td>
<td>665</td>
<td>0.71</td>
<td>737 K</td>
</tr>
<tr>
<td>Mars</td>
<td>149</td>
<td>0.17</td>
<td>213 K</td>
</tr>
</tbody>
</table>

### 2.1 Budyko’s 1-D Energy Balance Model

The following three exercises complement those from Session 4.

2. Recall Budyko’s 1-D equation from Session 4 as formulated by K.K. Tung [2]:

\[
R \frac{\partial T}{\partial t} = Q s(y)(1 - \alpha(y, \eta)) - (A + BT) - C(T - \bar{T}),
\]

where the first term of the right hand side represents the incoming solar radiation, the second term represents the outgoing longwave radiation (OLR), and the last term represents the meridional heat transport.

Let \( \bar{\alpha}(\eta) = \int_0^1 s(y)\alpha(y, \eta)dy \). You may have shown that the global average temperature at equilibrium is given by

\[
\bar{T}^* = \bar{T}^*(\eta) = \frac{1}{B}(Q(1 - \bar{\alpha}(\eta)) - A),
\]

and that equilibrium temperature profiles for Budyko’s equation (2.2) are given by

\[
T^* = T^*(y, \eta) = \frac{Q}{B + C} \left( s(y)(1 - \alpha(y, \eta)) + \frac{C}{B}(1 - \bar{\alpha}(\eta)) \right) - \frac{A}{B}.
\]

Assume the albedo function is given by

\[
\alpha(y, \eta) = \begin{cases} 
\alpha_w, & y < \eta \\
\alpha_i, & y > \eta.
\end{cases}
\]

(a) How does \( T^*(y, \eta) \) vary with the parameter \( A \)? Discuss in the context of the OLR term in the model equation.
Figure 2.1: The graph of $h(\eta)$

(b) Set $A = 202, B = 1.9, C = 3.04, Q = 343, \alpha_w = 0.32, \alpha_i = 0.62$, and $s(y) = 1.241 - 0.723y^2$. How does the shape of $T^*(y, \eta)$ vary with changes in the parameter $C$? Investigate by plotting $T^*(y, \eta)$ for several $C$-values in MATLAB. Interpret these changes in the context of the role $C$ plays in modeling meridional heat transport.

3. Recall the temperature at equilibrium at the ice line is defined to be

$$T^*(\eta) = \frac{1}{2} \left( \lim_{y \to \eta^-} T^*(y, \eta) + \lim_{y \to \eta^+} T^*(y, \eta) \right), \quad (2.5)$$

and that for albedo function (2.4) you showed that

$$T^*(\eta) = \frac{Q}{B+C} \left( s(\eta)(1 - \alpha_0) + \frac{C}{B}(1 - \alpha(\eta)) \right) - \frac{A}{B}, \quad \alpha_0 = \frac{1}{2}(\alpha_w + \alpha_i). \quad (2.6)$$

(a) In the climate modeling literature ice is assumed to melt if the temperature is below the critical temperature $T_c = -10^\circ C$. Let $h(\eta)$ denote the difference between the temperature at equilibrium at the ice line $\eta$ and $T_c$, that is, set $h(\eta) = T^*(\eta) - T_c$, with $T^*(\eta)$ given in equation (2.6).

Set the parameter values and $s(y)$ to be as in Exercise 2.

(i) Use MATLAB to plot the function $h(\eta)$ to find two ice line positions $\eta_1 < \eta_2$ satisfying $T^*(\eta_i) = -10^\circ C$, $i = 1, 2$, as in the figure below.

(ii) Plot the corresponding equilibrium solutions $T^*(y, \eta_i), \ i = 1, 2$.

(b) Consider the graph in Figure 2.1. What happens to the graph of $h(\eta)$ as the parameter $A$ varies? Numerically determine the value of $A_0$ for which the curve $h(\eta)$ is tangent to the line $T_c = -10^\circ C$. What can you say about equilibrium solutions $T^*(y, \eta)$ satisfying $T^*(\eta) = -10^\circ C$ in a small interval of $A$-values centered at $A_0$?
The relationship between $A$ and $\eta$. This exercise fits in Sessions 4 and 6. Suppose we require the temperature at the ice line at equilibrium to equal $T_c$, that is, $h(\eta) = 0$, where $h(\eta)$ is given as in Exercise 3.

(a) Deduce the following relationship between the parameter $A$ and the position $\eta$ of the ice line at equilibrium

$$A(\eta) = \frac{B}{B+C} \left( Q s(\eta)(1 - \alpha_0) + \frac{C}{B} Q(1 - \sigma(\eta)) \right) - B T_c. \quad (2.7)$$

(b) If Earth is in an ice free state, it will remain in an ice free state provided the temperature at equilibrium at the north pole is greater than $T_c$. Show that this will be the case provided

$$A < B \left( \frac{Q}{B+C} \left( s(1)(1 - \alpha_w) + \frac{C}{B} \left( 1 - \int_0^1 \alpha_w s(y)dy \right) \right) - T_c \right).$$

(c) If Earth is in the snowball state, it will remain completely ice covered provided the temperature at equilibrium at the equator is less than $T_c$. Show that this will be the case provided

$$A > B \left( \frac{Q}{B+C} \left( s(0)(1 - \alpha_i) + \frac{C}{B} \left( 1 - \int_0^1 \alpha_i s(y)dy \right) \right) - T_c \right).$$

(d) Suppose $s(y)$, $\alpha(y, \eta)$ and the parameters (other than $A$), are as in Exercise 2. Use parts (a)-(c) to generate the bifurcation diagram below. What does the model say about the position of the ice line at equilibrium if the CO$_2$ concentration is very high or very low?

\begin{itemize}
  \item High CO$_2$
  \item Low CO$_2$
\end{itemize}

(e) Describe the behavior of the model as $A$ increases through the bifurcation value at roughly $A = 211$. Compare with Exercise 3(b).
2.2 Dimension Reduction of Budyko’s EBM

5. This exercise fits in Session 4 and is adapted from a note by R. McGehee and E. Widiasih [9]. From the dynamical systems perspective, the state space of the Budyko equation

\[ \frac{\partial T}{\partial t} = Q_s(y)(1 - \alpha(y, \eta)) - (A + BT) - C(T - \bar{T}) \]  

(2.8)

is the space of functions giving the temperature profile \( T \) as a function of \( y \). Therefore, the state space is infinite dimensional. In this exercise we consider a finite dimensional version of this equation.

a. Show that the equilibrium solution of Budyko’s equation is piecewise quadratic, assuming the insolation function \( s(y) \) is quadratic, the albedo function \( \alpha(y, \eta) \) is the piecewise constant function

\[ \alpha(y, \eta) = \begin{cases} 
\alpha_1, & y < \eta \\
\alpha_2, & y > \eta \\
\frac{\alpha_1 + \alpha_2}{2}, & y = \eta,
\end{cases} \]

and the ice line \( \eta \) is fixed.

b. We may write the temperature profile \( T(y) \) as a piecewise function, with a potential discontinuity at \( \eta \), as

\[ T(t, y) = T(y) = \begin{cases} 
U(t, y), & y < \eta \\
V(t, y), & y > \eta.
\end{cases} \]

Write the following expressions in terms of \( U(t, y) \), \( V(t, y) \), and \( \eta \):

i. \( T(\eta) = \int_0^1 T(y) dy \)

ii. \( T(\eta) \) (assuming that the ice line temperature is the average of the left and the right hand limits of the temperature profile)

iii. Show that \( \frac{\partial}{\partial t} T(t, y) \) can be written as a piecewise continuous equation:

\[ R \frac{\partial}{\partial t} U(t, y) = Q_s(y)(1 - \alpha_1) - (A + BT) - C(T - \bar{T}), \quad 0 \leq y < \eta \]  

(2.9)

\[ R \frac{\partial}{\partial t} V(t, y) = Q_s(y)(1 - \alpha_2) - (A + BT) - C(T - \bar{T}), \quad \eta < y \leq 1. \]

c. Because of the symmetry assumption, the distribution function \( s(y) \) is assumed to be even as a function of \( y \), and hence the same assumption will hold for \( U \) and \( V \). It is convenient to use the first two even Legendre polynomials

\[ p_0 = 1 \quad \text{and} \quad p_2(y) = \frac{1}{2}(3y^2 - 1). \]

We represent \( U(t, y) \) and \( V(t, y) \) and \( s(y) \) in terms of these polynomials:
\[ U(t, y) = u_0(t)p_0(y) + u_2(t)p_2(y), \quad (2.10a) \]
\[ V(t, y) = v_0(t)p_0(y) + v_2(t)p_2(y), \quad (2.10b) \]
\[ s(y) = s_0p_0(y) + s_2p_2(y). \quad (2.10c) \]

Show that by equating the coefficients of the Legendre polynomials, system (2.9) results in the following system of four ordinary differential equations of the Legendre polynomial coefficients:

\[
R\dot{u}_0 = Q(1 - \alpha_1) - A - (B + C)u_0 - CT(\eta) \quad (2.11)
\]
\[
R\dot{v}_0 = Q(1 - \alpha_2) - A + (B + C)v_0 - CT(\eta)
\]
\[
R\dot{u}_2 = Qs_2(1 - \alpha_1) - (B + C)u_2
\]
\[
R\dot{v}_2 = Qs_2(1 - \alpha_2) - (B + C)v_2.
\]

d. Consider the change of variables:

\[
w = \frac{u_0 + v_0}{2}, \quad z = u_0 - v_0.
\]

Using this change of variables, show that three of the equations in system (2.11) decouple, resulting in the following system:

\[
R\dot{w} = Q(1 - \alpha_0) - A - (B + C)w + CT \quad (2.12)
\]
\[
R\dot{z} = Q(\alpha_2 - \alpha_1) - (B + C)z
\]
\[
R\dot{u}_2 = Qs_2(1 - \alpha_1) - (B + C)u_2
\]
\[
R\dot{v}_2 = Qs_2(1 - \alpha_2) - (B + C)v_2.
\]

e. The decoupled equations can now be solved separately. Solve the last three equations for \( z(t) \), \( u_2(t) \), \( v_2(t) \) and find the limiting behaviors of these functions as \( t \) goes to infinity.

Based on this exercise, it can be shown there is a globally attracting invariant manifold for system (2.12), and on this manifold the equations reduce to

\[
R\dot{w} = Q(1 - \alpha_0) - A - (B + C)w + CT(\eta), \quad (2.13)
\]

where

\[
T(\eta) = w + \frac{Q(\alpha_2 - \alpha_1)}{B + C} \left( \eta - \frac{1}{2} + \frac{s_2}{2} (\eta^3 - \eta) \right).
\]

f. Coupling with the ice line.
We have so far treated Budyko’s equation in terms of the temperature profile by assuming the ice line $\eta$ is fixed. The following equation is one way to model the ice line movement

$$\dot{\eta} = \epsilon(T(\eta) - T_c),$$

(2.14)

where $0 < \epsilon \ll 1$.

Provide a possible explanation for the use of this equation in the model.

g. Collecting the work we have done in this exercise, we conclude that Budyko’s temperature profile equation coupled with the ice line may be represented as the following 2-D system of equations:

$$\begin{align*}
R\dot{w} &= Q(1 - \alpha_0) - A - (B + C)w + C\bar{T}(\eta) \\
\dot{\eta} &= \epsilon(T(\eta) - T_c),
\end{align*}$$

(2.15)

where it can be shown that

$$T(\eta) = w + \frac{Q(\alpha_2 - \alpha_1)}{B + C} \left( \eta - \frac{1}{2} + \frac{s_2}{2}(\eta^3 - \eta) \right)$$

and

$$T(\eta) = T(w, \eta) = w + \frac{Qs_2(1 - \alpha_0)}{B + C}p_2(\eta).$$

Draw the nullclines of system (2.15), find the two equilibria, and determine their respective stability types.

h. Simulate the ice line and the global average temperature for the following initial values. You may find the following MATLAB files helpful.

* bud_main_45.m
* bud_sys.m

i. The invariant curve. Solve system (2.15) for $w$ when $\epsilon = 0$, and let $w = \Phi_0(\eta)$ represent this solution.

j. The system (2.15) of two equations for $w$ and $\eta$ may be reduced even further to just one equation in $\eta$, due to the separation of time scale between the two variables (see [14], for example). The following is the 1-D reduced system:

$$\dot{\eta} = \epsilon h(\eta),$$

(2.16)

where

$$h(\eta) = \Phi_0(\eta) + \frac{Qs_2(1 - \alpha_0)}{B + C}p_2(\eta) - T_c.$$ 

Simulate $\eta(t)$ and investigate the behavior of $\eta$ as $Q$ is allowed to change.
2.3 Black Body Radiation

The exercises in this section complement Session 6.

6. Assuming the Earth radiates like a blackbody with all OLR escaping to space, the
global mean temperature can be modeled by the equation

\[ R \frac{dT}{dt} = Q(1 - \alpha) - \sigma T^4, \quad (2.17) \]

as seen previously. The atmosphere can be incorporated into this simple model by
including an OLR \textit{emissivity factor} \( \epsilon \):

\[ R \frac{dT}{dt} = Q(1 - \alpha) - \epsilon \sigma T^4. \]

Note \( \epsilon = 1 \) yields an atmosphere completely transparent to OLR, as in equation (2.17).
With \( Q = 343 \text{ W/m}^2 \), \( \alpha = 0.3 \) and \( \sigma = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4} \), what value of \( \epsilon \) gives a global mean temperature \( T^* = 15.4^\circ \text{C} \) at equilibrium (close to our current
average temperature)?

7. A blackbody with temperature \( T \) emits radiation at frequency \( \nu \) with an intensity
given by Planck’s Law

\[ B(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}, \]

where \( h = 6.625 \times 10^{-34} \text{ J s} \) (Planck’s constant), \( k = 1.37 \times 10^{-23} \text{ J K}^{-1} \) (Boltzman’s
constant), \( c = 3 \times 10^8 \text{ m s}^{-1} \) (speed of light), \( \nu \) is the frequency of radiation in Hz,
and \( T \) is the temperature in kelvins.

(a) Recall the amount of radiant energy \( \Delta F(\nu) \) within a frequency interval \( (\nu, \nu + \Delta\nu) \)
that will flow through a given increment of area \( \Delta A \) within a solid angle \( \Delta \Omega \) of a
particular direction in a time interval \( \Delta t \) is

\[ \Delta F(\nu) = I(\nu) \cos \theta \Delta \Omega \Delta A \Delta\nu \Delta t, \]

where \( I = I(\nu) \) is the intensity. Using the relation \( \nu = cn \), show that the intensity \( I_n \) as a function of wavenumber \( n \) is \( I_n = cI(cn) \). Conclude that, as a function of
wavenumber and temperature, Planck’s Law can be written \( B_n(n, T) = cB(cn, T) \).

(b) Use the result in part (a) to generate the plot below (note the units!).
8. The average surface temperatures of Venus and Mars are 737 K and 213 K, respectively. As we’ve seen, the blackbody temperature at which a planet radiates energy is

\[ T = \frac{1}{\sqrt{2}} (1 - \alpha)^{1/4} \sqrt{\frac{T_{\text{sun}}}{d}} T_{\text{sun}}, \tag{2.18} \]

assuming the planet’s atmosphere is transparent to the energy emitted by its surface. Given the data below, compute the radiating temperature \( T \) for each of Venus and Mars via equation (2.18). Why might your computed value for Venus be so much lower than 737 K? Why might your computed value for Mars be so close to 213 K?

<table>
<thead>
<tr>
<th></th>
<th>Sun</th>
<th>albedo ( \alpha )</th>
<th>distance from sun ( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( T_{\text{sun}} = 5770 ) K</td>
<td>Venus</td>
<td>0.71</td>
</tr>
<tr>
<td></td>
<td>( r_{\text{sun}} = 6.9599 \times 10^9 ) km</td>
<td>Mars</td>
<td>0.17</td>
</tr>
</tbody>
</table>

2.4 The Milankovitch Cycle

The exercises in this section fits in Session 8.

9. We saw in lecture that the incoming solar radiation is quasiperiodic, affected by the eccentricity, the obliquity and the precession of Earth’s orbit. As the Budyko energy balance equation pertains to an annual average temperature, however, it is independent of the effect of precession, which is seasonal. In this exercise, we explore the ice line evolution \( \eta(t) \) from the previous exercise, as its insolation variables \( Q \) and \( s_2 \), are forced by computational data of the orbital elements (from Laskar, et al). We further compare the result to observational data of \( \delta^{18}O \) from ocean sediments, which is a proxy for ice volume.

Full simulations of these exercises using MATLAB Graphical User Interface can be found in the MATLAB file `PaleoBudyko.m`. 
a. The computation of the dependence of $Q$ on eccentricity $e$ and of $s(y)$ on $\beta$ can be found in the paper by McGehee and Lehman [10]:

\begin{align}
Q(e) &= \frac{Q_0}{1 - e^2}, \\
s(y, \beta) &= 1 + s_2(\beta)p_2(y), \text{ where} \\
s_2(\beta) &= \frac{5}{16}(-2 + 3\sin^2(\beta)).
\end{align}

(2.19a) (2.19b) (2.19c)

Obtain Laskar’s computational data for eccentricity $e$, obliquity $\beta$ from the website http://www.imcce.fr/Equipes/ASD/insola/earth/earth.html. Notice the data are given in 1,000 year increments. Compute the insolation parameters $Q$ and $s_2$ in yearly increment based on equations (2.19).

Notes: 1. You can also obtain the precession data from the website above.
2. The instructions to convert the data to a MATLAB format can be found in the MATLAB introduction section.
3. You may need to use some interpolation scheme to convert the temporal increment.

b. Simulate $\eta(t)$ using equation (2.16) and the forcings (2.19). Then plot the result over the past 5 million years.

c. Obtain the observational data, known in the geoscience circles as the Lisiecki Raymo 2005 $\delta^{18}O$ stack, from Lisiecki-Raymo website: http://lorraine-lisiecki.com/stack.html. Notice the temporal interval of this stack is not regular, so you should use an interpolation scheme to render the data in yearly increments.

Use the following formula to convert $\delta^{18}O$ to $\eta$ as derived in [10]:

\[ \delta^{18}O = 3.2 - 11.7(\eta - 0.92). \]

d. By this point, you should have two types of arrays for $\eta$: first from your computation using the reduced Budyko’s equation $\dot{\eta}$ forced by Laskar’s data and, secondly, from converting the Lisiecki-Raymo stack.

Plot the two $\eta$’s on the same time axis for the period available in the $\delta^{18}O$ stack (about 5.2 million years ago to today).

What do you notice? How close are the two data sets to agreement? In what time period? Suggest an explanation for your observations.

2.5 Continuous Albedo Functions and A Contraction Theorem

The exercises in this section complement Sessions 4 and 6.
10. (a) A discrete time approach to Budyko’s equation (2.2) was introduced recently by E. Widiasih [3]. Given an ice line position \( \eta \) and an initial temperature distribution \( T_0(y, \eta) \), set

\[
T_{n+1}(y, \eta) = T_n(y, \eta) + F(T_n(y, \eta)),
\]

where

\[
F(T) = \frac{K}{R} \left( Q_s(y)(1 - \alpha(y, \eta)) - (A + BT) - C(T - \int_0^1 Tdy) \right).
\]

The constant \( K \) is the number of seconds in one year. Equation (2.20) can be interpreted then as Euler’s method applied to equation (2.2) with step size one year, with \( T_n(y, \eta) \) denoting the temperature at latitude \( y \) at year \( n \).

Let \( \eta = 0.2 \).

(i) Plot the equilibrium solution \( T^*(y, 0.2) \) (see equation (2.3)).

(ii) Letting \( T_0(y, 0.2) = 39 - 10y^2 \), use MATLAB and recurrence (2.20) to reproduce the plot below.

(iii) Let \( T_0(y, 0.2) = -100^\circ C \). Compute and plot \( T_n(y, 0.2) \), with \( n \) large enough so as to deduce the limiting behavior of the sequence \( \{T_n(y, 0.2)\} \).

This exercise evidently indicates that the equilibrium solution \( T^*(y, 0.2) \) is attracting in some appropriate function space.

11. (a) A continuous approximation to the albedo function given in equation (2.4) is

\[
\alpha(y, \eta) = \frac{\alpha_i + \alpha_w}{2} + \frac{\alpha_i - \alpha_w}{2} \cdot \tanh M(y - \eta),
\]

where \( \alpha_w \) and \( \alpha_i \) are the albedos of open water and ice, respectively. The parameter \( M \) controls the abruptness of the transition from open ocean to ice. Let \( \alpha_w = 0.32, \alpha_i = 0.62, M = 25 \).

(i) Plot \( \alpha(y, 0.2), \alpha(y, 0.6) \) and \( \alpha(y, 0.9) \).
(ii) With the remaining parameters and \(s(y)\) as in Exercise 4, plot \(T^*(y, \eta)\) for \(\eta = 0.2, 0.6, 0.9\) using albedo function (2.21). Note the equilibrium solutions are now continuous at the ice line \(\eta\), with corresponding temperature value \(T^*(\eta, \eta) \equiv T^*(\eta)\).

(b) Consider the mapping introduced in equation (2.20), albeit with the albedo function given in equation (2.21). Let \(T_0(y, 0.2) = 39 - 10y^2\). By plotting \(T_n(y, 0.2)\) as \(n\) increases, investigate the long term behavior of the sequence \(\{T_n(y, 0.2)\}\). Repeat for \(T_0(y, 0.2) = -100^\circ\text{C}\). Note: This exercise requires non-trivial programming in MATLAB.

(c) Consider albedo function (2.21), again with parameters as in part (a) above. Note the temperature at equilibrium at the ice line is now given by

\[
T^*(\eta) = \frac{Q}{B + C} \left( s(\eta)(1 - \alpha(\eta, \eta)) + \frac{C}{B} (1 - \bar{\alpha}(\eta)) \right) - \frac{A}{B}. \quad (2.22)
\]

By plotting the function \(h(\eta) = T^*(\eta) - T_c\), show there exist \(0 < \eta_1 < \eta_2 < 1\) with \(h(\eta_i) = 0, i = 1, 2\). Thus, we again have two equilibrium solutions for which the temperature at the ice line equals \(T_c = -10^\circ\text{C}\).

12. Exercises 10 and 11 provide evidence for the attractive nature of the equilibrium solution \(T^*(y, 0.2)\) of Budyko’s equation (2.2). Note the ice line remained stationary in all simulations while the temperatures \(T_n(y, 0.2)\) either decreased to \(T^*(y, 0.2)\) or increased to \(T^*(y, 0.2)\). In the former case one would expect the ice line to move equatorward as the planet cools down, while in the latter case the ice line should move poleward as the planet warms up, as discussed in Hands-On Session 4.

The Budyko-Ice line model [3] includes an equation for the movement of the ice line:

\[
R \frac{\partial T}{\partial t} = Qs(y)(1 - \alpha(y, \eta)) - (A + BT) - C(T - T) \quad (2.23)
\]

\[
\frac{d\eta}{dt} = \epsilon(T(\eta) - T_c),
\]

where \(\epsilon\) is a positive constant and \(T(\eta) = T(\eta, \eta)\).

(a) What do equilibrium solutions to system (2.23) look like? In what space do they live?

(b) Let \(\alpha(y, \eta)\) be given as in equation (2.21). Given certain conditions on the parameters, and for sufficiently small \(\epsilon\), E. Widiasih [3] showed that system (2.23) has a locally attracting 1-dimensional invariant curve on which the dynamics are governed by the ODE

\[
\frac{d\eta}{dt} = \epsilon h(\eta), \quad (2.24)
\]

with \(h(\eta)\) as in Exercise 11. By referring to the plot of \(h(\eta)\) generated in Exercise 11, answer the following questions.
(i) For $\eta \in (\eta_2, 1)$, note $h(\eta) < 0$. What can you conclude about the movement of the ice line as temperature profile–ice line pairs evolve according to system (2.23)?

(ii) For $\eta \in (\eta_1, \eta_2)$, note $h(\eta) > 0$. What can you say about the movement of the ice line in this case? What can you subsequently conclude about the stability of the small ice cap corresponding to $\eta_2$?

(iii) For $\eta \in (0, \eta_1)$, note $h(\eta) < 0$. What can you say about the movement of the ice line in this case? What can you conclude about the stability of the large ice cap corresponding to $\eta_1$?

(iv) Again referring to your plot of $h(\eta)$, what happens if the Earth is ever ice free? What if the Earth is ice covered? Discuss in the context of model (2.23).

### 2.6 Budyko’s Model and the Jormungand Albedo Function

This section complements Sessions 4 and 6.

13. The albedo function in the Jormungand model of the extreme Neoproterozoic glacial episodes developed by Abbot et al. in [4] is given by

$$
\alpha(y, \eta) = \begin{cases} 
\alpha_w, & y < \eta \\
\frac{1}{2}(\alpha_w + \alpha_2(\eta)), & y = \eta \\
\alpha_2(y), & y > \eta,
\end{cases}
$$

(2.25)

where $\alpha_2(y) = \frac{1}{2}(\alpha_s + \alpha_i) + \frac{1}{2}(\alpha_s - \alpha_i) \tanh M(y - 0.35)$. In this model $\alpha_w$ is the albedo of open water, $\alpha_i$ is the albedo of bare sea ice, and $\alpha_s$ is the albedo of snow covered ice. The model assumes sea ice acquires a snow cover only for latitudes above $y = 0.35$.

Set $Q = 321, A = 170, B = 1.5, C = 2.25, \alpha_w = 0.35, \alpha_i = 0.45, \alpha_s = 0.8$, and $M = 25$. Use MATLAB for the following investigations.

(a) Plot $\alpha(y, \eta)$ for $\eta = 0.1, 0.2, 0.3, 0.6$.

(b) Plot the Jormungand equilibrium solutions $T^*(y, \eta)$, $\eta = 0.1, 0.2, 0.3$ and 0.6, with $T^*(y, \eta)$ as in equation (2.3) but with albedo function (2.25). Note each equilibrium temperature distribution has a discontinuity at the ice line $\eta$. Why should the planet be significantly colder at equilibrium in the Jormungand model than it was using albedo function (2.4) or albedo function (2.21)?

(c) Note that we have defined $\alpha(\eta, \eta)$, the albedo at the ice line, via equation (2.25). The temperature at the ice line at equilibrium $T^*(\eta)$ is then given by equation (2.22), albeit using the Jormungand albedo function (2.25). Once again, set $h(\eta) = T^*(\eta) - T_c = T^*(\eta) + 10$. Reproduce the plot below, from which we see the existence of three equilibrium solutions to Budyko’s equation which satisfy $T^*(\eta) = -10^\circ C$. 

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(d) In a brief essay, and following in spirit the instructions in Exercise 12(b), determine which of the large ice cap at $\eta_1$, the medium ice cap at $\eta_2$, and the small ice cap at $\eta_3$ is stable and which is unstable. What happens in the Jormungand model if the Earth is ice free? What if the Earth is in a snowball state? Compare and contrast with the behavior of Budyko’s model with albedo function (2.4).

(e) Note the existence of two values $A_0 < A_1$ at which the graph of $h(\eta)$ will be tangent to the line $T_c = -10^\circ$C. Numerically determine $A_0$ and $A_1$, and describe the changes in the set $\{\eta : h(\eta) = -10\}$ in neighborhoods of each of these $A$-values.

14. In this exercise you will draw a bifurcation diagram for the Jormungand model. We are interested in the ways in which the equilibrium solutions for which the temperature at the ice line is $-10^\circ$C change as the parameter $A$ varies.

Set $T^*(\eta) = -10^\circ$C, with $T^*(\eta)$ as in Exercise 13. Solving the resulting equation for $A = A(\eta)$ yields

$$A(\eta) = \frac{B}{B + C} \left( Q_s(\eta)(1 - \alpha(\eta,\eta)) + \frac{C}{B} Q(1 - \sigma(\eta)) \right) - BT_c, \quad (2.26)$$

as in Exercise 4.

(a) With the parameters (other than $A$) as in Exercise 13, and with the Jormungand albedo function (2.25), generate the bifurcation plot below.
(b) Describe the behavior of the model in each of the following scenarios.
   (i) As \( A \) decreases through the bifurcation value at roughly \( A = 159 \).
   (ii) As \( A \) increases through the bifurcation value at roughly \( A = 172 \).
   (iii) As \( A \) increases through the bifurcation value at roughly \( A = 180 \).

2.7 Further Exploration: Coupling \( \eta \) and \( A \)

This section complements Session 9.

15. In the above exercises we treated the greenhouse gas parameter \( A \) as fixed. In this exercise we consider \( A \) as a variable allowed to evolve with time. The background story of this exercise centers on the snowball Earth events which occurred roughly 600 million years ago. As we saw in Exercise 12, a snowball state is eternal, if all the parameters remain unchanged.

The geologist Joseph Kirschvink proposed a theory on how the Earth escaped a snowball state as follows: Earth has a long term carbon cycle that depends on the weathering process, which is a carbon sink, and volcanic \( \text{CO}_2 \) output, a carbon source. When the climate is too cold (due to lower latitudinal continents, for example), ice cover advances to the equator, shutting down the long term carbon sinks, while volcanoes still spew out carbon dioxide. This allows for a \( \text{CO}_2 \) build-up in the atmosphere and, eventually, there is enough greenhouse gas effect to warm up the planet. Once the planet warms up enough so that there is some open ocean, the ice albedo feedback effect takes place, and deglaciation happens at a relatively faster speed than that at which ice advances [2, pages 135-138].

(a) Based on Kirschvink’s idea, write a differential equation \( \frac{dA}{dt} \) to model the evolution of \( A \). Use the bifurcation diagram in Exercise 4 as a starting point. Since
you have to deal with an ice line at the equator and the pole, that is, the boundaries for \( \eta \), you may have to write separate equations for \( \eta \) in the boundary. **Hint:** Start with a simple equation for \( A' \) that depends on \( \eta \).

(b) What behavior do you expect from the two dimensional \( A - \eta \) system? Is there a possibility of any cycle?

(c) Notice the usual fundamental theorem of existence and uniqueness of solutions, hence the classical notion of solutions, does not apply to this system due to the discontinuity of the vector field at the boundary. One may wish to study the system using numerical simulation. Write the differential equations that may be used in a MATLAB simulation.

16. We repeat Exercise 15 for the Jormungand world.

(a) Again, based on Kirschvink's idea, write a differential equation \( \frac{dA}{dt} \) to model the evolution of \( A \). Use the bifurcation diagram in Exercise 14 as a starting point. As before, since you have to deal with an ice line at the equator and the pole, that is, the boundaries for \( \eta \), you need to write separate equations for \( \eta \) in the boundary (\( \eta = 0 \) and \( \eta = 1 \)).

(b) What behavior do you expect from the two dimensional \( A - \eta \) system in this Jormungand world? Describe any difference between the Jormungand world in Exercise 14 and the Budyko world in Exercise 4.

### 2.8 Additional Exercises

17. (Appropriate for Session 4) Let \( \theta \) denote latitude, let \( y = \sin \theta \), and suppose \( T(t, y) \) is a zonally averaged temperature profile which is assumed to be symmetric across the equator, as in Budyko’s model. Let \( r \) denote the radius of the planet.

(a) Show that \( y \) represents the percentage of the planet’s surface between latitudes \( \arcsin y \) and \( \arcsin(-y) \).

(b) Given latitudes \( 0 \leq \theta_1 < \theta_2 \leq \frac{\pi}{2} \), show the area of the surface between \( \theta_1 \) and \( \theta_2 \) is \( 2\pi r^2 (\theta_2 - \theta_1) \cos \beta \) for some \( \beta \in (\theta_1, \theta_2) \).

(c) Show that the global average temperature is given simply by \( \bar{T} = \int_0^1 T(t, y) \, dy \).

18. (Appropriate for Session 4) This problem concerns the ice free and snowball Earth states vis-à-vis Budyko’s equation \((2.2)\). Set the parameter values and \( s(y) \) to be as in Exercise 4.

(a) Suppose the Earth is in an ice free state, so that for all \( y \) we have \( \alpha(y, \eta) = \alpha(y) = 0.32 \). Compute \( \bar{\pi} \) in this case, and plot the corresponding equilibrium temperature profile \( T^*(y) \). Will ice ever form in this scenario?

(b) Suppose the Earth is in a snowball state, so that for all \( y \) we have \( \alpha(y, \eta) = \alpha(y) = 0.62 \). Compute \( \bar{\pi} \) in this case, and plot the corresponding equilibrium temperature profile \( T^*(y) \). Will ice ever melt in this scenario?
Further Explorations Using XPP and XPPAUT

In this section, we will explore the integration and bifurcation/continuation tool XPPAUT, available at http://www.math.pitt.edu/bard/xpp/xpp.html [13]. To run the free software, you will need a running X server. For Windows, Xming is a good option. For Mac users, X11 should already be installed on your computer, and if not, it is available for free in the App store. After getting the X server running, follow the download and installation instructions on the website listed above. I've found that for Mac users, the easiest way to do this is simply to click DOWNLOAD and choose the most up-to-date package. Installation instructions are in the package.

1. Revisiting Stefan-Boltzman There are many examples to play with in the ode folder within the XPPAUT package. This is where you will save all of your own .ode files. Let’s create one now. Consider the differential equation from Hands On Session 2 with the Stefan-Boltzman law for black-body radiation:

\[
\frac{dT}{dt} = Q(1 - \alpha) - \sigma T^4
\]  
(2.27)

(a) Creating a .ode file Note that when solving for equilibria, one finds two solutions— one positive and one negative. We should be able to see these both in XPP and in AUTO, and we will find that only on of the solutions is stable.

Open a plain text editor and copy and paste the following code into it:

```
# 0DEBM.ode
temp'=(Q*(1-alpha)-(sigma*temp^4))
param alpha=0.3,sigma=0.001,Q=343
@total=200,ylo=-50,yhi=50,xlo=-200,xhi=200,bounds=10000,
dt=0.05,maxstor=10000
done
```

Save the code as 0DEBM.ode in your ode file folder. It is important that your file extension is .ode, and not ode.txt. Click, drag and drop the .ode file on the XPP icon. This should open XPP.

(b) Playing with XPP On the XPP interface, choose Integrate then initial Conditions (capital letters are hot keys). What do you see? Do you think this solution is stable? If so, is it always stable? XPP also allows for backward-in-time integration. Choose nUmerics then Dt and add a negative sign. Investigate the stability of the second solution.

i. In the .ode file, a large value of \( \sigma \), the radiation constant, has been chosen for ease of numerics. What happens to the solutions as \( \sigma \to 0 \)? Investigate this in XPP by changing \( \sigma \) in the Parameter window, found at the top of the XPP interface.

This section was prepared by Anna Barry
(c) **Calling AUTO** In order to call the bifurcation software AUTO, your XPP data needs to have settled on an equilibrium solution. You can check for this by clicking the **Data** icon at the top of the XPP interface and paging through the data. If the temperature appears to remain constant, then your solution has converged. Close the data window and click **File, Auto**. A new window will open.

(d) **Continuation of Equilibria** Click the **Parameter** button and set Par1 to alpha. This will be our bifurcation parameter. Next, Click **Axes** and **hi-lo**. This choice will show the highest and lowest values of the solutions, and is particularly useful for visualizing periodic orbits, which is beyond the scope of this tutorial. Choose reasonable axes and values of Xmin, Ymin, Xmax, and Ymax. This defines your plot window. We will not need to mess with **Numerics** yet, so hit **Run, Steady state**. Interpret what you see.

(e) **Stability** Notice the circle in the lower left corner of the AUTO interface. This displays the Floquet Multiplier for the solution at a given numbered point. These numbered points and their stability can be investigated using the **Grab** option. Hit **Grab**, and a large cross should appear over data point 1. The tab button takes you to data point 2. Continuing to hit the tab button moves you through all of the data points. Which are stable? What type of bifurcation do you see, and where does it occur? Can you give physical or mathematical reasons for this?

2. **Revisiting the $\eta - A$ system** Recall the equation for the motion of the ice line

$$\frac{d\eta}{dt} = \varepsilon h(\eta)$$

where

$$h(\eta) = \frac{Q}{B+C} \left( s(\eta)(1 - a_0) + \frac{C}{B} (1 - \bar{\alpha}(\eta)) \right) - \frac{A}{B} - T_c$$

(a) **Creating a more complex .ode file** Suppose we make a simple choice for the evolution of $A$. For instance, take

$$\frac{dA}{dt} = \delta \varepsilon (\eta - \eta_c)$$

Can you give any physical reason as to why this linear equation might be a reasonable choice? Another parameter $\delta$ has been introduced, allowing for the assumption that $A$ evolves on a different timescale than $\eta$. Let’s assume $\delta$ is smaller than the timescale of $\eta$ (in particular, one could choose $0 < \delta < 1$). We can absorb the parameter $\varepsilon$ into the timescale. Create an .ode file for the two-dimensional $\eta - A$ system, or use the one below.

```plaintext
# nA.ode
n'=(Q/(B+C)*(s(n)*(1-alpha0)+C/B*(1-alphabar(n))))-A/B-Tc
A'=del*(n-nc)
param Q=343,B=1.9,C=3.04,alpha0=.47,Tc=-10,nc=0.62633,del=1
```

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s(n) = 1 - 0.482/2*(3*n^2 - 1)
alphabar(n) = 0.32*n + 0.62*(1 - n)
init n = 0.8 A = 202
@ total = 200, ylo = 0, yhi = 1, xlo = 0, xhi = 200, bounds = 10000, dt = 0.01,
maxstor = 100000
done

Explore the dynamics of nA.ode in XPP. Vary $\eta_c$. Are there any periodic orbits?

**Optional Challenge** Note: this is very nontrivial. You’ve noticed that when $\eta_c$ reaches a certain value the dynamics seem to “blow-up.” Is this really what is happening? Modify your XPP .ode file to restrict $\eta$ to the interval [0, 1]. You may have to introduce some special functions, such as the Heaviside function.

(b) **Exploring the $\eta - A$ phase space.** Erase your plot and change your viewing window so that $A$ is on the horizontal axis and $\eta$ is on the vertical axis. Once you’ve adjusted your window, hit **Nullclines, New.** See anything familiar? Check out the **Dir. field.** Play with initial conditions and parameters to explore the behavior of the system. If your graphs get too burdened, you can always **Erase** and start again.

(c) **Creating a bifurcation diagram for the fast subsystem** Let’s assume that $A$ is a “slow” variable in this slow fast system. If we set $\delta = 0$ then we arrive back at the ice line dynamics, and $A$ becomes a parameter again. Here, we know from previous exercises how $\eta$ looks as a function of $A$, but let’s now recreate this bifurcation diagram in AUTO. You may want to use the .ode code below.

```plaintext
# nAfastsubsystem.ode
n'=(Q/(B+C)*(s(n)*(1-alpha0)+C/B*(1-alphabar(n))))-A/B-Tc
param Q=343,B=1.9,C=3.04,alpha0=.47,Tc=-10,A=202
s(n)=1-0.482/2*(3*n^2 - 1)
alphabar(n)=.32*n+.62*(1-n)
init n = 0.9
@ total = 200, ylo = 0, yhi = 1, xlo = 0, xhi = 100, bounds = 10000, dt = 0.01,
maxstor = 10000
done

Once you’ve settled on an equilibrium point in XPP, call AUTO as before. Create a good viewing window with $A$ as the main parameter. Hint: In order to create a full bifurcation diagram, you may need to open the **Numerics** window and allow $dt$ to be negative at some point.

(d) **Saving AUTO data and importing into XPP** Finally, you’ll create a .dat file which can be imported into XPP, MATLAB, Mathematica, and other numerical software. Once you’ve created a bifurcation diagram in AUTO, select **File, Write points.** You can save the file in the form .dat in a directory of your choice (as long as you know how to find it).

(e) **Importing bifurcation data into XPP** Next, close XPPAUT and open nA.ode. Adjust the window as it was when you were creating your bifurcation diagram using **View axes, 2d.** To import the bifurcation diagram into XPP, choose **Graphic stuff, Freeze, Bif.Diag** and then select the bifurcation diagram you
would like to import. Now you can plot solutions on top of the bifurcation dia-
gram. Try increasing the value of $\delta$ significantly and increasing or decreasing $\eta_c$.
Do you see any oscillatory behavior? Can you explain it? **Optional: the ca-
nard explosion** The last question above may have required some knowledge of
the behavior of slow-fast systems. If you are familiar with such systems, explain
why increasing $\delta$ results in more visible oscillations. Do these oscillations still
exist for smaller $\delta$? How do you know? Can you find any? Why or why not?
Bibliography


