CHAPTER 17

Hyperbolic Transformations

> Though the text of your article on 'Crystal Symmetry and Its Generalizations' is much too learned for a simple, self-made pattern man like me, some of the text-illustrations and especially Figure 7, page 11, gave me quite a shock . . .
>
> If you could give me a simple explanation how to construct the following circles, whose centres approach gradually from the outside till they reach the limit, I should be immensely pleased and very thankful to you! Are there other systems besides this one to reach a circle limit?
>
> Nevertheless I used your model for a large woodcut (of which I executed only a sector of 120 degrees in wood, which I printed 3 times). I am sending you a copy of it.
>
> – M. C. Escher (1898–1972), from a letter to H. S. M. Coxeter, as reported in [5]

The illustration that gave Escher "quite a shock" was a drawing that Coxeter had produced of a regular tiling of the Poincaré disk by triangles. This tiling inspired Escher to create his "Circle Limit I" woodcut. A version of this image created by Doug Dunham is shown in Figure 17.1, with a few of the basic triangle tiles outlined in bold arcs. For more on Dunham's work on Escher-like hyperbolic tilings, see [7].

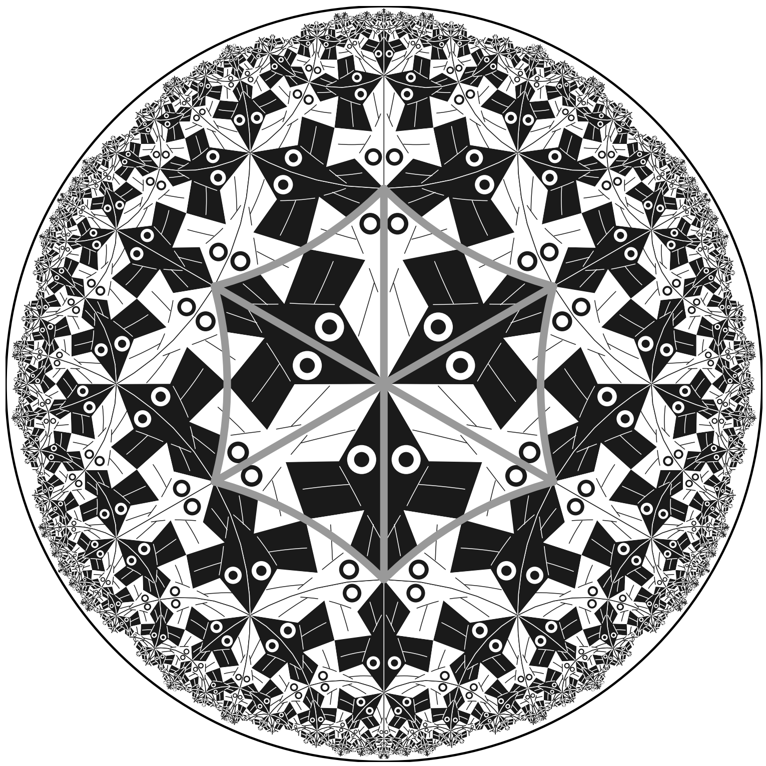


Figure 17.1 Dunham's Version of Circle Limit I

The tiling illustrated in this figure is a *regular* tiling—all triangles used to build the tiling are congruent via *hyperbolic* transformations; that is, one-to-one and onto functions of the Poincaré disk to itself that preserve the Poincaré distance function.

In Chapter 5 we saw that the group of distance-preserving functions of the Euclidean plane consisted of Euclidean reflections, rotations, translations, and glide reflections. Such isometries not only preserved lengths of segments, but preserved angles as well. We also saw that the set of all Euclidean isometries formed an algebraic structure called a *group*. Before we study the nature of hyperbolic isometries, we will look at an alternate way to represent Euclidean isometries that will be useful in defining hyperbolic isometries.

Each element of the group of Euclidean isometries can be represented by a complex function. For example, the function $f(z) = z + (v_1 + iv_2)$ represents translation by the vector $v = (v_1, v_2)$, and $g(z) = e^{i\phi}z$ represents a rotation about the origin by an angle ϕ. Rotations about other points in the plane, say rotation about a by an angle of ϕ, can be defined by a sequence of isometries: first, translation by the vector $-a$, then rotation by ϕ about the origin, and then translation by the vector a. Thus, the desired rotation can be represented as $h(z) = (e^{i\phi}(z-a))+a = e^{i\phi}z + (a - e^{i\phi}a)$.

Thus, any Euclidean translation or rotation can be represented as a complex function of the form

$$f(z) = e^{i\phi}z + b \tag{17.1}$$

with b complex and ϕ real. These are the orientation-preserving isometries of the Euclidean plane. The set of all such functions, which are sometimes called *rigid motions* of the plane, forms a group called the *Euclidean group*.

What are the analogous orientation- and distance-preserving functions in hyperbolic geometry? In particular, what are the orientation- and distance-preserving functions in the Poincaré model? Since all rigid Euclidean isometries can be realized as certain one-to-one and onto complex functions, a good place to look for hyperbolic transformations might be in the *entire* class of one-to-one and onto complex functions.

But, which functions should we consider? Since we are concentrating on the Poincaré model, we need to find one-to-one and onto orientation-preserving functions that preserve the Euclidean notion of angle, but do not preserve Euclidean length. In section 16.1, we studied functions that preserved angles and preserved the *scale* of Euclidean lengths *locally*. Such functions were called *conformal* maps. Euclidean rigid motions such as rotations and translations preserve angles, and preserve length *globally*. Such motions comprise a subset of all conformal maps.

If we consider the entire set of all conformal maps of the plane onto itself, then by Theorem 16.5 such maps must have the form $f(z) = az+b$, where $a \neq 0$ and b is a complex constant. Since $a = |a|e^{i\phi}$, then f is the composition of a translation, a rotation, and a scaling by $|a|$. Thus, f maps figures to *similar* figures. The set of all such maps forms a group called the group of *similitudes* or *similarity transformations* of the plane. If $b = 0$ ($f(z) = az, a \neq 0$), we call f a *dilation* of the plane. Most similarity transformations cannot be isometries of the Poincaré model since most similarities (like translations and scalings) do not fix the boundary circle of the Poincaré disk.

Clearly, we must expand our set of possible transformations. One way to do this is to consider the set of all one-to-one and onto conformal maps of the *extended* complex plane to itself. In Theorem 16.6 we saw that these maps have the form

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0 \tag{17.2}$$

17.1 MÖBIUS TRANSFORMATIONS

Definition 17.1. *A* Möbius transformation *is a function on the extended complex plane defined by equation 17.2. The set of Möbius transformations forms a group called the* Möbius group*.*

Every Möbius transformation is composed of simpler transformations.

Theorem 17.1. *Let T be a Möbius transformation. Then T is the composition of translations, dilations, and inversion ($g(z) = \frac{1}{z}$).*

Proof: If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is the composition of a translation with a dilation.

If $c \neq 0$, then $f(z) = \frac{a}{c} - \frac{ad-bc}{c^2}\frac{1}{z+\frac{d}{c}}$. Thus, f is the composition of a translation (by $\frac{d}{c}$), an inversion, a dilation (by $-\frac{ad-bc}{c^2}$), and a translation (by $\frac{a}{c}$). □

Note that the Möbius group includes the group of Euclidean rigid motions ($|a| = 1, c = 0, d = 1$), and the group of similarities ($a \neq 0, c = 0, d = 1$) as subgroups. Also note that we could define Möbius transformations as those transformations of the form in equation 17.2 with $ad - bc = 1$, by dividing the numerator by an appropriate factor.

Within the group of Möbius transformations, can we find a subgroup that will serve as the group of orientation-preserving isometries for the Poincaré model? We shall see that there is indeed such a sub-group. Before we can prove this, we need to develop a toolkit of basic results concerning Möbius transformations.

17.1.1 Fixed Points and the Cross Ratio

How many fixed points can a Möbius transformation f have? Suppose $f(z) = z$. Then

$$z = \frac{az+b}{cz+d}$$

So, $cz^2 + (d - a)z - b = 0$. This equation has at most two roots. Thus, we have

Lemma 17.2. *If a Möbius transformation f has three or more fixed points, then $f = id$, where id is the identity Möbius transformation.*

We saw in Chapter 5 that an isometry is uniquely defined by its effect on three non-collinear points. For Möbius transformations we can relax the condition on collinearity.

Theorem 17.3. *Given any three distinct complex numbers z_1, z_2, z_3, there is a unique Möbius transformation f that maps these three values to a specified set of three distinct complex numbers w_1, w_2, w_3.*

Proof: Let $g_1(z) = \frac{z-z_2}{z-z_3}\frac{z_1-z_3}{z_1-z_2}$. Then g_1 is a Möbius transformation (the proof is an exercise) and g_1 maps z_1 to 1, z_2 to 0, and z_3 to the point at infinity.

Let $g_2(w) = \frac{w-w_2}{w-w_3}\frac{w_1-w_3}{w_1-w_2}$. We see that g_2 is a Möbius transformation mapping w_1 to 1, w_2 to 0, and w_3 to ∞.

Then $f = g_2^{-1} \circ g_1$ will map z_1 to w_1, z_2 to w_2, and z_3 to w_3.

Is f unique? Suppose f' also mapped z_1 to w_1, z_2 to w_2, and z_3 to w_3. Then $f^{-1} \circ f'$ has three fixed points, and so $f^{-1} \circ f' = id$ and $f' = f$.
□

Corollary 17.4. *If two Möbius transformations f, g agree on three distinct points, then $f = g$.*

Proof: This is an immediate consequence of the preceding theorem.
□

The functions g_1 and g_2 used in the proof of Theorem 17.3 are called *cross ratios*.

Definition 17.2. *The cross ratio of four complex numbers z_0, z_1, z_2, and z_3 is denoted by (z_0, z_1, z_2, z_3) and is the value of*

$$\frac{z_0 - z_2}{z_0 - z_3}\frac{z_1 - z_3}{z_1 - z_2}$$

The cross ratio is an important invariant of the Möbius group.

Theorem 17.5. *If z_1, z_2, and z_3 are distinct points and f is a Möbius transformation, then $(z, z_1, z_2, z_3) = (f(z), f(z_1), f(z_2), f(z_3))$ for any z.*

Proof: Let $g(z) = (z, z_1, z_2, z_3)$. Then $g \circ f^{-1}$ will map $f(z_1)$ to 1, $f(z_2)$ to 0, and $f(z_3)$ to ∞. But, $h(z) = (z, f(z_1), f(z_2), f(z_3))$ also maps $f(z_1)$ to 1, $f(z_2)$ to 0, and $f(z_3)$ to ∞. Since $g \circ f^{-1}$ and h are both Möbius transformations, and both agree on three points, then $g \circ f^{-1} = h$. Since $g \circ f^{-1}(f(z)) = (z, z_1, z_2, z_3)$ and $h(f(z)) = (f(z), f(z_1), f(z_2), f(z_3))$, the result follows. □

17.1.2 Geometric Properties of Möbius Transformations

Of particular interest to us will be the effect of a Möbius transformation on a circle or line.

Definition 17.3. *A subset of the plane is a* cline *if it is either a circle or a line.*

The cross ratio can be used to identify clines.

Theorem 17.6. *Let z_0, z_1, z_2, and z_3 be four distinct points. Then the cross ratio (z_0, z_1, z_2, z_3) is real if and only if the four points lie on a cline.*

Proof: Let $f(z) = (z, z_1, z_2, z_3)$. Then since f is a Möbius transformation, we can write

$$f(z) = \frac{az + b}{cz + d}$$

Now $f(z)$ is real if and only if

$$\frac{az + b}{cz + d} = \frac{\overline{a}\overline{z} + \overline{b}}{\overline{c}\overline{z} + \overline{d}}$$

Multiplying this out, we get

$$(a\overline{c} - c\overline{a})|z|^2 + (a\overline{d} - c\overline{b})z - (d\overline{a} - b\overline{c})\overline{z} + (b\overline{d} - d\overline{b}) = 0 \tag{17.3}$$

If $(a\bar{c}-c\bar{a})=0$, let $\alpha=(a\bar{d}-c\bar{b})$ and $\beta=b\bar{d}$. Equation 17.3 simplifies to

$$Im(\alpha z+\beta)=0$$

This is the equation of a line (proved as an exercise).

If $(a\bar{c}-c\bar{a})\neq 0$, then dividing through by this term we can write equation 17.3 in the form

$$|z|^2+\frac{a\bar{d}-c\bar{b}}{a\bar{c}-c\bar{a}}z-\frac{d\bar{a}-b\bar{c}}{a\bar{c}-c\bar{a}}\bar{z}+\frac{b\bar{d}-d\bar{b}}{a\bar{c}-c\bar{a}}=0$$

Let $\gamma=\frac{a\bar{d}-c\bar{b}}{a\bar{c}-c\bar{a}}$ and $\delta=\frac{b\bar{d}-d\bar{b}}{a\bar{c}-c\bar{a}}$. Since $a\bar{c}-c\bar{a}$ is pure imaginary, we have that

$$\bar{\gamma}=(-)\frac{d\bar{a}-b\bar{c}}{a\bar{c}-c\bar{a}}=\frac{d\bar{a}-b\bar{c}}{c\bar{a}-a\bar{c}}$$

Equation 17.3 becomes

$$|z|^2+\gamma z+\overline{\gamma z}+\delta=0.$$

Or,

$$|z+\bar{\gamma}|^2=-\delta+|\gamma|^2$$

After multiplying and regrouping on the right, we get

$$|z+\bar{\gamma}|^2=\left|\frac{ad-bc}{a\bar{c}-c\bar{a}}\right|^2$$

Since $ad-bc\neq 0$, this gives the equation of a circle centered at -$\bar{\gamma}$.

□

Theorem 17.7. *A Möbius transformation f will map clines to clines. Also, given any two clines c_1 and c_2, there is a Möbius transformation f mapping c_1 to c_2.*

Proof: Let c be a cline and let z_1, z_2, and z_3 be three distinct points on c. Let $w_1=f(z_1)$, $w_2=f(z_2)$, and $w_3=f(z_3)$. These three points will lie on a line or determine a unique circle. Thus, w_1, w_2, and w_3 will lie on a cline c'. Let z be any point on c different than z_1, z_2, or z_3. By the previous theorem we have that (z,z_1,z_2,z_3) is real. Also, $(f(z),w_1,w_2,w_3)=(f(z),f(z_1),f(z_2),f(z_3))=(z,z_1,z_2,z_3)$, and thus $f(z)$ is on the cline through w_1, w_2, and w_3.

For the second claim of the theorem, let z_1, z_2, and z_3 be three distinct points on c_1 and w_1, w_2, and w_3 be three distinct points on c_2. By Theorem 17.3 there is a Möbius transformation f taking z_1, z_2, z_3 to w_1, w_2, w_3. It follows from the first part of this proof that f maps all points on c_1 to points on c_2. □

So, Möbius transformations map circles to circles. They also preserve *inversion* through circles. Recall from Chapter 2 that the inverse of a point P with respect to a circle c centered at O is the point P' on the ray $\overrightarrow{OP}$ such that $(OP')(OP) = r^2$, where r is the radius of c and OP is the hyperbolic length of $\overline{OP}$ (Figure 17.2).

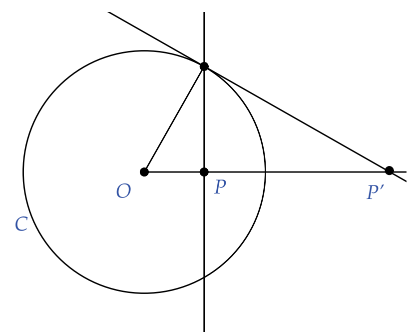


Figure 17.2 Circle Inversion

If $z = P$ and $a = O$, then the defining equation for the inverse z^* of z with respect to a circle c with radius r is

$$|z^* - a||z - a| = r^2$$

Since z^* is on the ray through a and z, we get that $z^* - a = r_1 e^{i\theta}$ and $z - a = r_2 e^{i\theta}$, and $|z^* - a||z - a| = r_1 * r_2 = r_1 e^{i\theta} r_2 e^{-i\theta} = (z^* - a)(\overline{z} - \overline{a})$. Thus,

$$z^* - a = \frac{r^2}{\overline{z} - \overline{a}} \tag{17.4}$$

It turns out that inversion can also be defined using the cross ratio.

Lemma 17.8. *Let z_1, z_2, and z_3 be distinct points on a circle c. Then, z^* is the inverse of z with respect to c if and only if $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$.*

Proof: Let c have center a and radius r. Then, since the cross ratio is invariant under translation by $-a$, we have

$$\begin{aligned}\overline{(z, z_1, z_2, z_3)} &= \overline{(z-a, z_1-a, z_2-a, z_3-a)} \\ &= (\overline{z-a}, \overline{z_1-a}, \overline{z_2-a}, \overline{z_3-a}) \\ &= (\overline{z}-\overline{a}, \frac{r^2}{z_1-a}, \frac{r^2}{z_2-a}, \frac{r^2}{z_3-a})\end{aligned}$$

Since the cross ratio is invariant under the transformation $f(z) = \frac{r^2}{z}$, we have

$$\overline{(z, z_1, z_2, z_3)} = (\frac{r^2}{\overline{z}-\overline{a}}, z_1-a, z_2-a, z_3-a)$$

Finally, translation by a yields

$$\overline{(z, z_1, z_2, z_3)} = (a + \frac{r^2}{\overline{z}-\overline{a}}, z_1, z_2, z_3)$$

So, if $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$, we see immediately that $z^* = a + \frac{r^2}{\overline{z}-\overline{a}}$ and z^* is the inverse to z.

On the other hand, if $z^* = a+\frac{r^2}{\overline{z}-\overline{a}}$, then $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$.
□

Definition 17.4. *Two points z and z^* are* symmetric with respect to a circle c *if* $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$ *for points z_1,z_2,z_3 on c.*

By the Lemma, this definition is not dependent on the choice of points z_1,z_2,z_3.

Theorem 17.9 (The Symmetry Principle)**.** *If a Möbius transformation f maps circle c to circle c', then it maps points symmetric with respect to c to points symmetric with respect to c'.*

Proof: Let z_1,z_2,z_3 be on c. Since

$$\begin{aligned}(f(z^*), f(z_1), f(z_2), f(z_3)) &= (z^*, z_1, z_2, z_3) \\ &= \overline{(z, z_1, z_2, z_3)} \\ &= \overline{(f(z), f(z_1), f(z_2), f(z_3))}\end{aligned}$$

the result follows. □

17.2 ISOMETRIES IN THE POINCARÉ MODEL

We now return to our quest of finding one-to-one and onto maps of the Poincaré disk to itself that are orientation-preserving and will preserve the Poincaré distance function. Our earlier idea was to search within the group of Möbius transformations for such functions. It is clear that any candidate Möbius transformation must map the Poincaré disk to itself and so must leave the boundary (unit) circle invariant.

Theorem 17.10. *A Möbius transformation f mapping $|z| < 1$ onto $|w| < 1$ and $|z| = 1$ onto $|w| = 1$ has the form*

$$f(z) = \beta \frac{z - \alpha}{\overline{\alpha} z - 1}$$

where $|\alpha| < 1$ and $|\beta| = 1$.

Proof: Let α be the point which gets sent to 0 by f. Then by equation 17.4, the inverse to $z = \alpha$, with respect to the unit circle, is the point $z^* = \frac{1}{\overline{\alpha}}$. By Theorem 17.9, this inverse point gets sent to the inverse of 0, which must be ∞.

If $\alpha \neq 0$, then $c \neq 0$ (∞ does not map to itself), and

$$f(z) = \left(\frac{a}{c}\right) \frac{z + \frac{b}{a}}{z + \frac{d}{c}}$$

Since α maps to 0, then $\frac{b}{a} = -\alpha$ and since $\frac{1}{\overline{\alpha}}$ gets mapped to ∞, then $\frac{d}{c} = -\frac{1}{\overline{\alpha}}$. Letting $\beta = \overline{\alpha}\left(\frac{a}{c}\right)$, we get

$$f(z) = \beta \frac{z - \alpha}{\overline{\alpha} z - 1}$$

Now $1 = |f(1)| = |\beta| \frac{|1-\alpha|}{|\overline{\alpha}-1|} = |\beta|$. Thus, $|\beta| = 1$.

If $\alpha = 0$, then $f(z) = \frac{a}{d} z + \frac{c}{d}$. Since $f(0) = 0$, $\frac{c}{d} = 0$, and since $|f(1)| = 1$, we get $|\beta| = |\frac{a}{d}| = 1$. □

Will transformations of the form given in this theorem preserve orientation and distance? Since such transformations are Möbius transformations, and thus conformal maps, they automatically preserve orientation. To determine if they preserve the Poincaré distance function, we need to evaluate the distance function for points represented as complex numbers.

Theorem 17.11. *The* hyperbolic distance *from z_0 to z_1 in the Poincaré model is given by*

$$\begin{aligned} d_P(z_0, z_1) &= |\ln((z_0, z_1, w_1, w_0))| \\ &= |\ln(\frac{z_0 - w_1}{z_0 - w_0}\frac{z_1 - w_0}{z_1 - w_1})| \end{aligned} \tag{17.5}$$

where w_0 and w_1 are the points where the hyperbolic line through z_0 and z_1 meets the boundary circle.

Proof: From our earlier development of the Poincaré model (section 7.2), we have

$$d_P(z_0, z_1) = |\ln(\frac{|z_0 - w_1|}{|z_0 - w_0|}\frac{|z_1 - w_0|}{|z_1 - w_1|})|$$

Since $|zw| = |z||w|$ and $|\frac{z}{w}| = \frac{|z|}{|w|}$, we have

$$d_P(z_0, z_1) = |\ln(|\frac{z_0 - w_1}{z_0 - w_0}\frac{z_1 - w_0}{z_1 - w_1}|)|$$

By Theorem 17.6, we know that $\frac{z_0-w_1}{z_0-w_0}\frac{z_1-w_0}{z_1-w_1}$, which is the cross ratio of z_0, z_1, w_1, and w_0, is real since all four points lie on a circle. Also, this cross ratio is non-negative (proved as an exercise). Thus,

$$d_P(z_0, z_1) = |\ln(\frac{z_0 - w_1}{z_0 - w_0}\frac{z_1 - w_0}{z_1 - w_1})|$$

□

Corollary 17.12. *Transformations of the form*

$$f(z) = \beta\frac{z - \alpha}{\overline{\alpha}z - 1}$$

where $|\alpha| < 1$ and $|\beta| = 1$ preserve the Poincaré distance function.

Proof: Let f be a transformation of the form described in the corollary, and let z_0, z_1 be two points in the Poincaré disk, with w_0, w_1 the points where the hyperbolic line through z_0, z_1 meets the boundary circle. Then, since f is a Möbius transformation, it will map clines to clines and will preserve angles. Thus, $f(z_0)$, $f(z_1)$ will be points in the

Poincaré disk and $f(z_0)$, $f(z_1)$, $f(w_0)$, $f(w_1)$ will all lie on a cline that meets the boundary circle at right angles. That is, these points will lie on a hyperbolic line. Also, since f maps the boundary to itself, we know that $f(w_0)$ and $f(w_1)$ will lie on the boundary.

Thus, by Theorem 17.5, we have

$$\begin{aligned} d_P(f(z_0), f(z_1)) &= |\ln((f(z_0), f(z_1), f(w_1), f(w_0)))| \\ &= |\ln(z_0, z_1, w_1, w_0))| \\ &= d_P(z_0, z_1) \end{aligned}$$

□

What types of transformations are included in the set defined by

$$f(z) = \beta \frac{z - \alpha}{\overline{\alpha} z - 1} \tag{17.6}$$

where $|\alpha| < 1$ and $|\beta| = 1$?

If $|\beta| = 1$, then $\beta = e^{i\theta}$. So, multiplication by β has the geometric effect of rotation about the origin by an angle of θ. Thus, if $\alpha = 0$ in equation 17.6, then $T(z) = -\beta z$ is a simple rotation about the origin by an angle of $\pi + \theta$.

On the other hand, if $\beta = 1$ in equation 17.6, consider the line $t\alpha$ passing through the origin and α. We have that $f(t\alpha) = \alpha \frac{t-1}{|\alpha|^2 - 1}$, which is again a point on the line through α. The map f can be considered a *translation* along this line.

Thus, we see that orientation- and distance-preserving maps contain rotations and translations, similar to what we saw in the Euclidean case. However, translations are not going to exhibit the nice parallel properties that they did in the Euclidean plane.

What about orientation-*reversing* isometries? Since the cross ratio appears in the distance function and is always real on Poincaré lines, then simple complex conjugation ($\overline{f}(z) = \overline{z}$) of Poincaré points will be a distance-preserving transformation in the Poincaré model and will reverse orientation.

In fact, $\overline{f}$, which is a Euclidean reflection, is also a hyperbolic reflection, as it fixes a hyperbolic line. Similarly, Euclidean reflection in any diameter will be a hyperbolic reflection. This is most easily seen by the fact that Euclidean reflection about a diameter can be expressed as the conjugation of $\overline{f}$ by a rotation R about the origin by $-\theta$, where θ is the angle the diameter makes with the x-axis. Since rotation about the origin is an isometry in the Poincaré model, then $R^{-1} \circ \overline{f} \circ R$ is also an isometry.

All other hyperbolic reflections about lines that are not diameters can be expressed as inversion through the circle which defines the line (proved as an exercise).

We can now determine the structure of the complete group of isometries of the Poincaré model. Let g be any orientation-reversing isometry of the Poincaré model. Then, $h = \overline{f} \circ g$ will be an orientation-preserving isometry, and so h must be a transformation of the type in equation 17.6. Since $\overline{f}^{-1} = \overline{f}$, we have that g can be expressed as the product of $\overline{f}$ with an orientation-preserving isometry. This same conjugation property would be true for *any* hyperbolic reflection. Thus, we have

Theorem 17.13. *The orientation-preserving isometries of the Poincaré model can be expressed in the form* $g = \beta\frac{z-\alpha}{\overline{\alpha}z-1}$*. Also, if* r *is hyperbolic reflection about some hyperbolic line, then all orientation-reversing isometries can be expressed as* $r \circ g$ *for some orientation preserving* g.

Hyperbolic isometries can be used to prove many interesting results in Hyperbolic geometry. Using Klein's Erlanger Program approach, in order to prove any result about general hyperbolic figures, it suffices to transform the figure to a "nice" position and prove the result there. For example, we can prove the following theorems on distance quite easily using this transformational approach.

Theorem 17.14. *Let* z *be a point in the Poincaré disk. Then*

$$d_H(0, z) = \ln\left(\frac{1+|z|}{1-|z|}\right)$$

Proof: Let $z = re^{i\theta}$ and T be rotation about the origin by $-\theta$. Then $d_H(0, z) = d_H(T(0), T(z)) = d_H(0, r)$. Now,

$$\begin{aligned} d_H(0, r) &= \left|\ln\left(\frac{0-1}{0-(-1)}\;\frac{r-1}{r-(-1)}\right)\right| \\ &= \left|\ln\left(\frac{1-r}{1+r}\right)\right| \\ &= \ln\left(\frac{1+r}{1-r}\right) \end{aligned}$$

Since $r = |z|$, the result follows. □

This result lets us convert from hyperbolic to Euclidean distance.

Corollary 17.15. *Let z be a point in the Poincaré disk. If $|z| = r$ and if δ is the hyperbolic distance from 0 to z, then*

$$\delta = \ln\left(\frac{1+r}{1-r}\right)$$

and

$$r = \frac{e^\delta - 1}{e^\delta + 1}$$

Proof: The first equality is a re-statement of the preceding theorem. The second equality is proved by solving for r in the first equality. □

Exercise 17.2.1. *Show that the set of Euclidean rigid motions $f(z) = e^{i\phi}z + b$, with b complex and ϕ real, forms a group.*

Exercise 17.2.2. *Show that the equation $Im(\alpha z + \beta) = 0$, with α and β complex constants, defines a line in the plane.*

Exercise 17.2.3. *Find a Möbius transformation mapping the circle $|z = 1|$ to the x-axis.*

The next four exercises prove that the set of Möbius transformations forms a group.

Exercise 17.2.4. *Let $f(z) = \frac{az+b}{cz+d}$, where $(ad - bc) \neq 0$, and let $g(z) = \frac{ez+f}{gz+h}$, where $(eh - fg) \neq 0$, be two Möbius transformations. Show that the composition $f \circ g$ is again a Möbius transformation.*

Exercise 17.2.5. *Show that the set of Möbius transformations has an identity element.*

Exercise 17.2.6. *Let $f(z) = \frac{az+b}{cz+d}$, where $(ad-bc) \neq 0$, be a Möbius transformation. Show that $f^{-1}(w)$ has the form $f^{-1}(w) = \frac{dw-b}{-cw+a}$ and show that f^{-1} is a Möbius transformation.*

Exercise 17.2.7. *Why does the set of Möbius transformations automatically satisfy the associativity requirement for a group?*

Exercise 17.2.8. *Show that the set of Möbius transformations that fix the unit circle is a group.*

Exercise 17.2.9. *Show that in the Poincaré model there is a hyperbolic isometry taking any point P to any other point Q. [Hint: Can you find an isometry taking any point to the origin?]*

Exercise 17.2.10. *Let $T(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$, with $\alpha \neq 0$. Show that T has two fixed points, both of which are on the unit circle. Thus, T has no fixed points in the Poincaré model of the hyperbolic plane. Why would it make sense to call T a translation?*

Exercise 17.2.11. *Show that the cross ratio term used in the definition of hyperbolic distance is always real and non-negative. [Hint: Use transformations to reduce the calculation to one that is along the x-axis.]*

Exercise 17.2.12. *Use the idea of conjugation of transformations to derive the formula for reflection across the diameter $y = x$ in the Poincaré model. [Hint: Refer to exercise 5.7.3.]*

Exercise 17.2.13. *In the definition of hyperbolic distance given by equation 17.5, we need to determine boundary points w_0 and w_1. Show that we can avoid this boundary calculation by proving that*

$$d_H(z_0, z_1) = \ln\left(\frac{|1 - z_0\overline{z_1}| + |z_0 - z_1|}{|1 - z_0\overline{z_1}| - |z_0 - z_1|}\right)$$

[Hint: Use the hyperbolic transformation $g(z) = \frac{z-z_1}{1-\overline{z_1}z}$ and the fact that d_H is invariant under g.]

Exercise 17.2.14. *Prove that if z_0, z_1, and z_2 are collinear in the Poincaré disk with z_1 between z_0 and z_2, then $d_P(z_0, z_2) = d_P(z_0, z_1) + d_P(z_1, z_2)$. This says that the Poincaré distance function is additive along Poincaré lines.*

Exercise 17.2.15. *Let l be a Poincaré line. Define a map f on the Poincaré disk by $f(P) = P'$, where P' is the inverse point to P with respect to the circle on which l is defined. We know by the results at the end of Chapter* **??** *that f maps the Poincaré disk to itself. Prove that f is an isometry of the Poincaré disk. Then, show that f must be a reflection. That is, inversion in a Poincaré line is a reflection in the Poincaré model. [Hint: Review the proof of Lemma 17.8.]*

17.3 ISOMETRIES IN THE KLEIN MODEL

At the end of the last section, we saw that reflections play an important role in describing the structure of isometries in the Poincaré model. This should not be too surprising. In Chapter 5 we saw that all Euclidean isometries could be built from one, two, or three reflections. The proof of this fact used only *neutral* geometry arguments. That is, the proof used no assumption about Euclidean parallels. Thus, any isometry in

Hyperbolic geometry must be similarly built from one, two, or three hyperbolic reflections.

In the Poincaré model, the nature of a reflection depended on whether the Poincaré line of reflection was a diameter or not. If the line of reflection was a diameter, then the hyperbolic reflection across that line was simple Euclidean reflection across the line. If the line was not a diameter, then hyperbolic reflection across the line was given by inversion of points through the circle that defined the line.

Let's consider the first class of lines in the Klein model. Since the Klein distance function is defined in terms of the cross ratio and since the cross ratio is invariant under complex conjugation and rotation about the origin, then these two transformations will be isometries of the Klein model. By the same argument that we used in the last section, we see that any Euclidean reflection about a diameter must be a reflection in the Klein model.

What about reflection across a Klein line that is not a diameter? Let's recall the defining properties of a reflection. By Theorem 5.6, we know that if P, P' are two Klein points, then there is a unique reflection taking P to P'. The line of reflection will be the perpendicular bisector of $\overline{PP'}$.

Thus, given a Klein line l in the Klein disk and given a point P, we know that the reflection of P across l can be constructed as follows. Drop a perpendicular line from P to l intersecting l at Q. Then, the reflected point P' will be the unique point on the ray opposite $\overrightarrow{QP}$ such that $\overline{PQ} \cong \overline{QP'}$. This point can be found by the following construction.

Theorem 17.16. *Let l be a Klein line that is not a diameter. Let P be a Klein point not on l. Let t be the Klein line through P that meets l at Q at right angles. Let $\overleftrightarrow{P\Omega}$ be the Klein line through P perpendicular to t, with omega point Ω. Let $\overleftrightarrow{Q\Omega}$ be the Klein line through Q and Ω that has ideal point $\Omega' \neq \Omega$. Finally, let P' be the point on t where the (Euclidean) line through the pole of t and Ω' passes through t. Then P' is the reflection of P across l (Figure 17.3).*

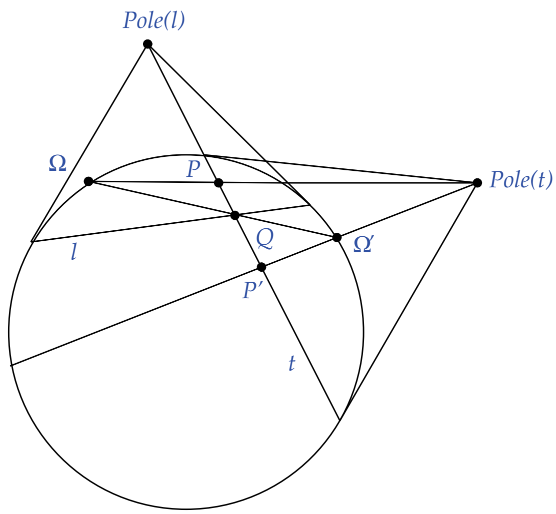


Figure 17.3

Proof: To construct t we draw a line from the pole of l through P. To construct $\overleftrightarrow{P\Omega}$ we draw a line from the pole of t through P. Both of these perpendiculars are guaranteed to exist by results from neutral geometry.

Now the line through Q and Ω must intersect the boundary at a point Ω' that is on the other side of t from Ω (in the Euclidean sense). Thus, the line through the pole of t and Ω' must intersect t at a point P'. If we can show that $\overline{PQ} \cong \overline{QP'}$, we are done.

Since $\overleftrightarrow{P'\Omega'}$ passes through the pole of t, then it is perpendicular to t (in the Klein sense). By Angle-Angle congruence of omega triangles (see exercise 7.3.10), we know that $\overline{PQ} \cong \overline{QP'}$. □

The construction described in the theorem is quite important for many other constructions in the Klein model. In the exercises at the end of this section, we will investigate other constructions based on this one.

We can use this theorem to show that any two Klein lines are congruent. That is, there is an isometry of the Klein model taking one to the other. To prove this result, we will need the following fact.

Lemma 17.17. *Given a Klein line l that is not a diameter of the Klein disk, we can find a reflection r that maps l to a diameter of the Klein disk.*

Proof: Let P be a point on l and let O be the center of the Klein disk. Let Q be the midpoint of $\overline{PO}$ (in the sense of Klein distance). Let n be the perpendicular to $\overline{PO}$ at Q. Then Klein reflection of l about n will map P to O and thus must map l to a diameter, since Klein reflections map Klein lines to Klein lines. □

Corollary 17.18. *Let l and m be Klein lines. Then there is an isometry in the Klein model taking l to m.*

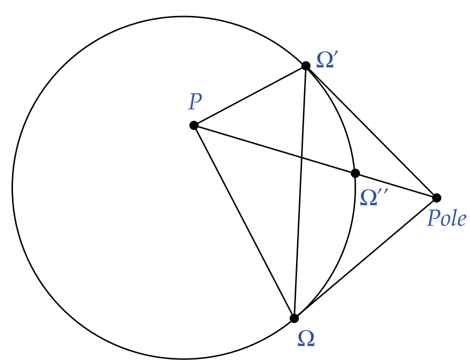
Proof: By the lemma we know there is a reflection r_l taking l to a diameter d_l. If d_l is not on the x-axis, let R_l be rotation by $-\theta$, where θ is the angle made by d_l and the x-axis. Then the Klein isometry $h_l = R_l \circ r_1$ maps l to the diameter on the x-axis. Likewise, we can find a Klein isometry h_m taking m to a diameter on the x-axis. Then $h_m^{-1} \circ h_l$ is a Klein isometry mapping l to m. □

Exercise 17.3.1. *Use the construction ideas of Theorem 17.16 to devise a construction for the perpendicular bisector of a Klein segment PP'.*

Exercise 17.3.2. *Devise a construction for producing a line l that is orthogonal to two parallel (but not limiting parallel) lines m and n.*

Exercise 17.3.3. *Show that in the Klein model there exists a pentagon with five right angles. [Hint: Start with two lines having a common perpendicular.]*

Exercise 17.3.4. *Let Ω and Ω' be two omega points in the Klein disk and let P be a Klein point. Let the Euclidean ray from the pole of $\Omega\Omega'$ through P intersect the unit circle at omega point Ω''. Show that $\overrightarrow{P\Omega''}$ is the Klein model angle bisector of $\angle \Omega P \Omega'$. [Hint: Use omega triangles.]*



Exercise 17.3.5. *Define a hyperbolic translation T in the Klein model as the product of two reflections r_l and r_m where l and m have a common perpendicular t. Show that t is invariant under T.*

Exercise 17.3.6. *Define a hyperbolic* parallel displacement D *in the Klein model as the product of two reflections* r_l *and* r_m *where* l *and* m *are limiting parallel to each other. Show that no Klein point* P *is invariant under* D*. [Hint: Assume that* P *is invariant (that is,* $r_l(P) = r_m(P)$*) and consider the line joining* P *to* $r_l(P)$*.]*

17.3.1 Mini-Project - The Upper Half-Plane Model

In this chapter we have looked in detail at the isometries of the Poincaré and Klein models of Hyperbolic geometry. Both of these models are based on the unit disk.

There is really nothing special about using the unit disk in these models. In the Poincaré model, for example, we could just as well have used any circle in the plane and defined lines as either diameters or arcs meeting the boundary at right angles. In fact, if c was any circle, we know there is a Möbius transformation f that will map the unit disk to c. Since Möbius transformations preserve angles, we could define new lines in c to be the image under f of lines in the Poincaré model. Likewise, circles in c would be images of Poincaré model circles, and the distance could be defined in terms of f as well.

A natural question to ask is whether there are models for hyperbolic geometry other than ones based on a set of points inside a circle.

If we think of the extended complex plane as being equivalent to the sphere via stereographic projection, then lines in the plane are essentially circles that *close up* at the point at infinity.

Is there a model for Hyperbolic geometry that uses as its boundary curve a Euclidean line? To build a model using a line as a boundary curve, we use the Klein Erlanger Program approach and determine the transformations that leave the line invariant. This is analogous to the work we did earlier to find the transformations that fix the unit circle in the Poincaré model. For simplicity's sake, let's assume our line boundary is the x-axis. Then we want to find Möbius transformations

$$f(z) = \frac{az + b}{cz + d}$$

that fix the x-axis.

Exercise 17.3.7. *Show that if f maps the x-axis to itself, then a, b, c, and d must all be real. [Hint: Use the fact that 0 and ∞ must be mapped to real numbers and that ∞ must also be the image of a real number.]*

What about the half-planes $y < 0$ and $y > 0$? Let's restrict our attention to those transformations that move points within one half-plane, say the upper half-plane $y < 0$.

Exercise 17.3.8. *Show that if f fixes the x-axis and maps the upper half-plane to itself, then $ad - bc > 0$. [Hint: Consider the effect of f on $z = i$.]*

We have now proved the following result.

Theorem 17.19. *If f fixes the x-axis and maps the upper half-plane to itself, then*

$$f(z) = \frac{az+b}{cz+d}$$

where a,b,c,d are all real and $ad - bc > 0$.

We will call the group of transformations in the last theorem the *Upper Half-Plane* group, denoted by U.

Exercise 17.3.9. *What curves should play the role of lines in the geometry defined by U? [Hint: Refer to Figure 17.4.]*

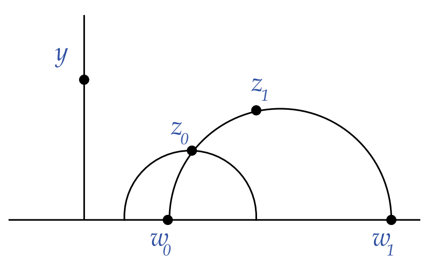


Figure 17.4 Upper Half-Plane Model

We can define distance just as we did for the Poincaré model in terms of the cross ratio. For example, in Figure 17.4 the distance from z_0 to z_1 will be defined as

$$d_U(z_0, z_1) = \ln((z_0, z_1, w_1, w_0))$$

Exercise 17.3.10. *What are the values of w_0 and w_1 in the distance formula if z_0 and z_1 are on a piece of a Euclidean line?*

Exercise 17.3.11. *Discuss why this geometry will satisfy the hyperbolic parallel postulate. [Hint: Argue that it is enough to show the postulate is satisfied for the y-axis and z_0 as shown in Figure 17.4.]*

The geometry defined by the group U will be another model for Hyperbolic geometry. This model (the upper half-plane model) is very similar to the Poincaré model. In fact, there is a conformal map taking one to the other. This map is defined by

$$g(z) = -i\frac{z+i}{z-i} \tag{17.7}$$

Exercise 17.3.12. *Show that g maps the unit circle onto the x-axis. [Hint: Consider $z = i, -i, 1$.]*

Since g is a Möbius transformation, then g will preserve angles. So, it must map Poincaré lines to upper half-plane lines. Also, since the distance function is defined in terms of the cross ratio and the cross ratio is invariant under g, then Poincaré circles will transform to upper half-plane circles.

We conclude that the upper half-plane model is *isomorphic* to the Poincaré model. Any property of one can be moved to the other by g or g^{-1}.

For future reference, we note that the inverse transformation to g is given by

$$g^{-1} = i\frac{w-i}{w+i} \tag{17.8}$$

17.4 WEIERSTRASS MODEL

There are other models of Hyperbolic geometry. One of the most interesting is the *Weierstrass model*. Here points are defined to be Euclidean points on one sheet of the *hyperboloid* $x^2+y^2-z^2 = -1$ (see Figure 17.5). Lines in this model are the curves on the top sheet of the hyperboloid that are created by intersecting the surface with planes passing through the origin. Each such line will be one branch of a hyperbola. For a complete development of this model, see [8].

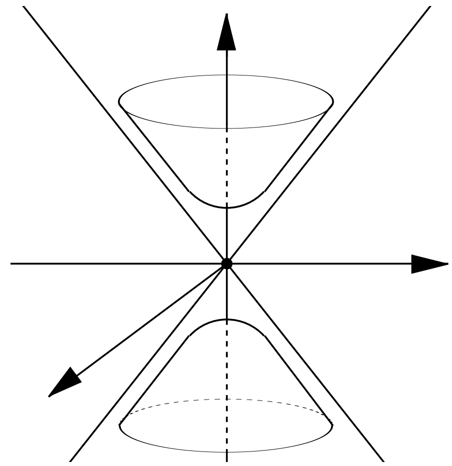


Figure 17.5 Weierstrass Model

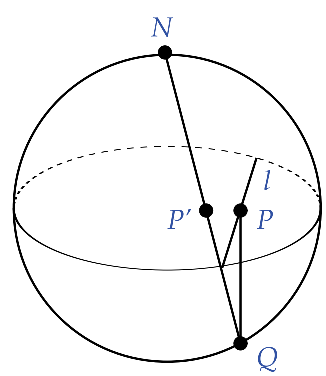
17.5 MODELS AND ISOMORPHISM

The two models of Hyperbolic geometry, the Poincaré model and the Klein model, are very similar. In fact, there is a conformal map taking one to the other that maps lines to lines, angles to angles, and the distance function of one model to the distance function of the other. The existence of such a function implies that the two models are *isomorphic* —they have identical geometric properties.

We will construct a one-to-one map from the Klein model to the Poincaré model as follows.

First, construct the sphere of radius 1 given by $x^2 + y^2 + z^2 = 1$. Consider the unit disk ($x^2 + y^2 = 1$) within this sphere to be the Klein disk.

Let N be the north pole of the sphere and let P be a point on the Klein line l as shown. Project P orthogonally downward to the bottom of the sphere, yielding point Q. Then, stereographically project from N, using the line from N to Q, yielding point P' in the unit disk.



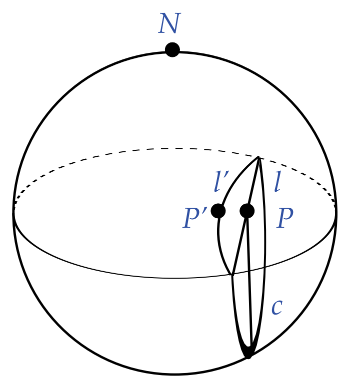
Define a function F from the Klein disk to the Poincaré disk by $F(P) = P'$, where P is a point in the Klein disk and P' is the unique point defined by the construction above. Since projection is one-to-one from the disk onto the lower hemisphere, and since stereographic projection is also one-to-one and onto from the lower hemisphere back to the unit disk, then the map F is a one-to-one, onto function from the Klein disk to the Poincaré disk.

The inverse to F, which we will denote by F', is the map that takes P' to P. That is, it projects P' onto the sphere and then projects this spherical point up to the disk, to point P. From our work in Chapter 16 on stereographic projection, we know that the equation for F' will be

$$F'(x, y) = (\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2})$$

How do the maps F and F' act on lines in their respective domains?

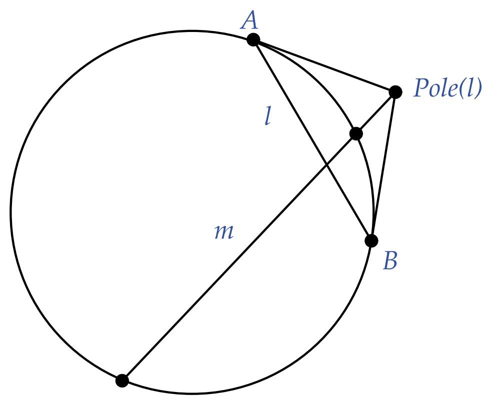
Let l be a Klein line. Projecting l orthogonally downward will result in a circular arc c on the sphere that meets the unit circle (equator) at right angles. Since stereographic projection preserves angles and maps circles to circles, stereographic projection of c back to the unit disk will result in a circular arc that meets the unit circle at right angles—a Poincaré line.



We see, then, that F maps Klein lines to Poincaré lines, *and* it preserves the ideal points of such lines, as the points where such lines meet the unit circle are not moved by F. Thus, F preserves the notions of *point* and *line* between the two models.

What about the notion of angle? Let's review the construction of a perpendicular in the Klein model. Recall that the *pole* of chord $\overline{AB}$ in a circle c is the inverse point of the midpoint of $\overline{AB}$ with respect to the circle. We defined a Klein line m to be perpendicular to a Klein line l based on whether l was a diameter of the Klein disk. If it is a diameter, then m is perpendicular to l if it is perpendicular to l in the Euclidean sense. If l is not a diameter, then m is perpendicular to l if the Euclidean line for m passes through the pole P of l.

Here are two lines $l = \overline{AB}$ and m that are perpendicular in the Klein model.



Since the pole of chord $\overline{AB}$ is also the intersection of the tangents at A and B to the circle, we see that the pole of $\overline{AB}$ will also be the *center* of the circle passing through A and B that is orthogonal to the unit circle. That is,

Lemma 17.20. *The pole of the Klein line $l = \overline{AB}$ will be the center of the orthogonal circle defining the Poincaré line $F(l)$.*

We can now prove that F preserves orthogonality between the models.

Theorem 17.21. *Two Klein lines l and m are perpendicular if and only if the corresponding Poincaré lines $F(l)$ and $F(m)$ are perpendicular.*

Proof: There are three cases to consider. First, suppose l and m are both diameters. Then, $F(l)$ and $F(m)$ are both diameters, and perpendicularity has the same (Euclidean) definition in both models.

Second, suppose l is a diameter and m is not and suppose that l and m are perpendicular. Then, the diameter l bisects chord m and thus passes through the pole of m. Now, $F(l) = l$ and since the center of $F(m)$ is the pole of l, $F(l)$ must pass through the center of $F(m)$. This implies that l is orthogonal to the tangent line to $F(m)$ at the point where it intersects $l = F(l)$ (Figure 17.6), and so $F(l)$ is orthogonal to $F(m)$. Reversing this argument shows that if $F(l)$ and $F(m)$ are orthogonal, then so are l and m.

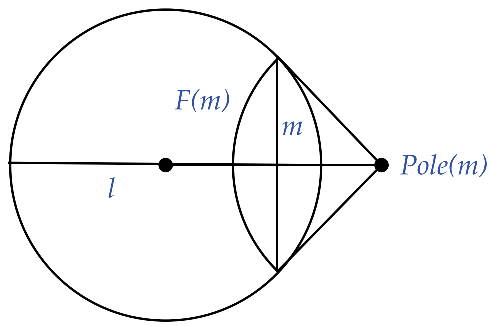


Figure 17.6

Last, suppose both l and m are not diameters of the unit circle. Then, l passes through the pole of m and vice versa. Also, these poles are the centers of $F(l)$ and $F(m)$ (Figure 17.7).

Suppose $F(m)$ is perpendicular to $F(l)$ (in the Poincaré sense). Let P and Q be the points where $F(m)$ meets the unit circle. Let c be the circle on which $F(l)$ lies and c' be the circle on which $F(m)$ lies. By Corollary 2.45, we know that inversion of c', and the unit circle, through c will switch P and Q. This is due to the fact that both c' and the unit circle are orthogonal to c, and so both will be mapped to themselves by inversion in c. Thus, P and Q are inverse points with respect to c, and the line through P and Q must go through the center of c (the pole of l). Then, m is perpendicular to l in the Klein sense.

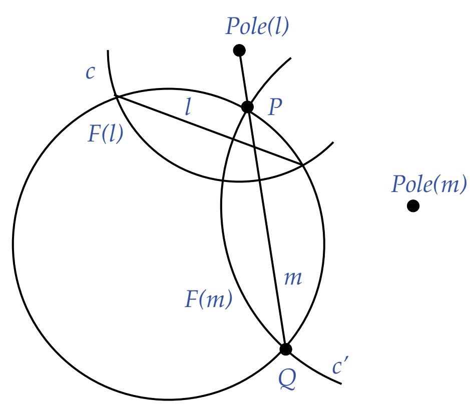


Figure 17.7

Conversely, if m is perpendicular to l in the Klein sense, then the (Euclidean) line for m intersects the pole of l (the center of circle c). Since the unit circle and c are orthogonal, the unit circle is mapped to itself by inversion in c. But, the inverse to P is the unique point that lies on the ray through the center of c and P (and on the unit circle) that gets mapped to another point on the unit circle. Thus, the inverse to P must be Q, and by Theorem 2.44 we have that c' is orthogonal to c, and the lines are perpendicular in the Poincaré sense. □

Since F preserves the definition of right angles between the Poincaré and Klein models, this gives us a natural way to define *all* angles in the Klein model: we will define the measure of a Klein angle to be the value of the Poincaré angle it corresponds to.

Definition 17.5. *Given three points A', B', C' in the Klein disk, the measure of the Klein angle defined by these points is the value of $\angle ABC$ in the Poincaré model, where $F(A') = A$, $F(B') = B$, and $F(C') = C$.*

With this definition, F is a one-to-one map from the Klein model to the Poincaré model that preserves points and lines. F will be an isomorphism between these models if we can show that F preserves the distance functions of the models.

To show this, we will borrow a couple of results from the next chapter on isometries in Hyperbolic geometry. Just as there are Euclidean isometries that will take any Euclidean line to the x-axis, so there are transformations in the Klein and Poincaré models that preseve the Klein and Poincaré distance functions and that map any line in these models to the diameter of the unit disk.

If we assume this property of the two models, then to show F preserves the distance functions of the models, it is enough to show that F preserves distances for hyperbolic lines that lie along the x-axis in the unit circle. Equivalently, we can show the inverse map F' preserves distances along such lines.

Theorem 17.22. *Let $P = (x_1, 0)$ and $Q = (x_2, 0)$ be two points in the Poincaré disk. Then*

$$d_P(P, Q) = d_K(F'(P), F'(Q))$$

Proof: By definition of d_P we have

$$d_P(P, Q) = \left|\ln(\frac{1+x_1}{1-x_1}\frac{1-x_2}{1+x_2})\right|$$

Also, since stereographic projection maps any diameter of the unit circle to itself, we have

$$d_K(F'(P), F'(Q)) = \frac{1}{2}\left|\ln(\frac{1+u_1}{1-u_1}\frac{1-u_2}{1+u_2})\right|$$

where $F'(P) = (u_1, 0)$ and $F'(Q) = (u_2, 0)$.

We know that $F'(x, 0) = (\frac{2x}{1+x^2}, 0)$. So,

$$\begin{aligned} 1 \pm u_1 &= 1 \pm \frac{2x_1}{1+x_1^2} \\ &= \frac{(1 \pm x_1)^2}{1+x_1^2} \end{aligned}$$

and similarly for u_2.

Thus, we get

$$\begin{aligned} d_K(F'(P), F'(Q)) &= \frac{1}{2}\left|\ln(\frac{\frac{(1+x_1)^2}{1+x_1^2}}{\frac{(1-x_1)^2}{1+x_1^2}}\frac{\frac{(1-x_2)^2}{1+x_2^2}}{\frac{(1+x_2)^2}{1+x_2^2}})\right| \\ &= \frac{1}{2}\left|\ln((\frac{1+x_1}{1-x_1}\frac{1-x_2}{1+x_2})^2)\right| \\ &= \frac{1}{2}2\left|\ln(\frac{1+x_1}{1-x_1}\frac{1-x_2}{1+x_2})\right| \\ &= \left|\ln(\frac{1+x_1}{1-x_1}\frac{1-x_2}{1+x_2})\right| \\ &= d_P(P, Q) \end{aligned}$$

□

We conclude that the map F is an isomorphism of the Klein and Poincaré models. That is, any geometric property valid in one of these models must be valid in the other model and vice versa.