Student Guide for *Exploring Geometry*

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Chapter 1

Geometry and the Axiomatic Method

The development of the axiomatic method of reasoning was one of the most profound events in the history of mathematics. In this chapter we explore axiomatic systems and their properties.

One strand running through the chapter is the search for the "ideal". The golden ratio is the ideal in concrete form, realized through natural and man-made constructions. Deductive reasoning from a base set of axioms is the ideal in abstract form, realized in the crafting of clear, concise, and functional definitions, and in the reasoning employed in well-constructed proofs.

Another strand in the chapter, and which runs through the entire text, is that of the interplay between the concrete and the abstract. As you work through this text, you are encouraged to "play" with concrete ideas, such as how the Golden Ratio appears in nature, but you are also encouraged to play (experiment) when doing proofs and more abstract thinking. The experimentation in the latter is of the mind, but it can utilize many of the same principles of exploration as you would use in a computer lab. When trying to come up with a proof you should consider lots of examples and ask "What if ...?" questions. Most importantly, you should interact with the ideas...
CHAPTER 1. AXIOMATIC METHOD

just as you interact with a computer lab project.

Student interaction with ideas and discovery of concepts is the primary organizing principle for the text. Interaction is encouraged in three ways. First, topics are introduced and developed in the text. Next, lab projects reinforce concepts, or introduce related ideas. Lastly, project results are discussed, and conclusions drawn in written lab reports. You will first read about concepts and hear them discussed in class. Then, you will conduct "experiments" to make the ideas concrete. Finally, you will conceptualize ideas by re-telling them in project reports.

The work you do in the lab and in group projects is a critical component of the course. The projects that are designed to be done in groups have an additional pedagogical advantage. You will find that by speaking with other students, using mathematical terms and concepts, you will better internalize such concepts and make them less abstract.

Notes on Lab Projects

The main difficulty you will face with the first lab project will be in learning the functionality of the Geometry Explorer program. One major point to watch out for is the notion of "attaching" objects together when doing their construction. For example, when you create a point on top of a line, the point becomes attached to the line. That is, when the point is moved it is constrained to follow the line.

In order to help with the formatting of lab reports there is a sample lab report for a "fake" lab on the Pythagorean Theorem in appendix A of this guide.
SOLUTIONS TO EXERCISES IN CHAPTER 1

Solutions to Exercises in Chapter 1

1.3 Project 1 - The Ratio Made of Gold

1.3.1 Since $AB = 2$, we have that $\frac{2}{x} = \frac{1+\sqrt{5}}{2}$. Solve for $x$ and clear the denominator of radicals.

1.3.3 Have some fun with this one, but do not get carried away with this idea and spend the whole class period on it!

1.4 The Rise of the Axiomatic Method

In this section we focus on reasoning in mathematics. The problems in this section may seem quite distant from the geometry you learned in high school, but the goal is to practice reasoning from the definitions and properties that an axiomatic system posits and then argue using just those basic ideas and relationships. This is good mental training. It is all too easy to argue from diagrams when trying to justify geometric statements.

1.4.1 If dictionaries were not circular, there would need to be an infinite number of different words in the dictionary.

1.4.3 Let a set of two different flavors be called a pairing. Suppose there were $m$ children and $n > m$ pairings. By Axiom 2 every pairing is associated to a unique child. Thus, for some two pairings $P_1, P_2$ there is a child $C$ associated to both. But this contradicts Axiom 3. Likewise, if $m > n$, then by Axiom 3 some two children would have the same pairing. This contradicts Axiom 2. So, $m = n$ and, since the number of pairings is $4 + 3 + 2 + 1 = 10$, there are 10 children.

1.4.5 There are exactly four pairings possible of a given flavor with the others. By exercise 1.4.3 we know that there are four distinct children associated to these pairings.

1.4.7 By Axioms 2 and 4 we have $ex = (xx^{-1})x = x(x^{-1}x)$. So all we need to do is show that $x^{-1}x = e$. Now, $(x^{-1}x)(x^{-1}x) = x^{-1}(xx^{-1})x = x^{-1}ex = x^{-1}x$, by Axioms 2, 3, and 4. Let $y = x^{-1}$. Then, $yy = y$ and $yyy^{-1} = yy^{-1}$ by Axiom 4. So, $y = e$ by Axiom
3 and 4 and the proof is complete. Note: This proof is a bit tricky – you may want to first experiment with $xx^{-1}$.

1.4.9 First we show that $1 \in M$. By Axiom 4 we know 1 is not the successor of any natural number. In particular, it cannot be the successor of itself. Thus, $1' \neq 1$ and $1 \in M$. Now, suppose $x \in M$. That is, $x' \neq x$. By Axiom 3 we have that $(x')' \neq x'$, and so $x' \in M$. Both conditions of Axiom 6 are satisfied and thus $M = N$.

1.4.11 Given $x$, let $M = \{y | x + y \text{ is defined}\}$. Then, by definition $1 \in M$. Suppose $y \in M$. Then, $x + y' = (x + y)'$ is defined and $y' \in M$. So, $M = N$ by Axiom 6. Now, since $x$ was chosen arbitrarily, addition is defined for all $x$ and $y$.

1.4.13 This is a good discussion question. Think about the role of abstraction versus application in mathematics. Think about how abstraction and application cross-fertilize one another.

1.5 Properties of Axiomatic Systems

This is a “meta” section. By this is meant that we are studying properties of axiomatic systems themselves, considering such systems as mathematical objects in comparison to other systems. This may seem quite foreign territory to you, but have an open mind and think about how one really knows that mathematics is true or logically consistent. We often think of mathematics as an ancient subject, but in this section we bring in the amazing results of the twentieth century mathematician Kurt Godel.

If this topic interests you, you may want to further research the area of information theory and computability in computer science. A good reference here is Gregory Chaitin’s book *The Limits of Mathematics* (Springer, 1998.) Additionally, much more could be investigated as to the various philosophies of mathematics, in particular the debates between platonists and constructionists, or between intuitionists and formalists. A good reference here is Edna Kramer’s *The Nature and Growth of Modern Mathematics* (Princeton, 1981), in particular Chapter 29 on Logic and Foundations.
SOLUTIONS TO EXERCISES IN CHAPTER 1

1.5.1 Let $S$ be the set of all sets which are not elements of themselves. Let $P$ be the proposition that “$S$ is an element of itself.” And consider the two propositions $P$ and the negation of $P$, which we denote as $¬P$. Assume $P$ is true. Then, $S$ is an element of itself. So, $S$ is a set which by definition is not an element of itself. So, $¬P$ is true. Likewise, if $¬P$ is true then $P$ is true. In any event we get $P$ and $¬P$ both true, and the system cannot be consistent.

1.5.3 Good research books for this question are books on the history of mathematics. This could be a good final project idea.

1.5.5 Let $P$ be a point. Each pairing of a point with $P$ is associated to a unique line. There are exactly three such pairings.

1.5.7 Yes. The lines and points satisfy all of the axioms.

1.5.9 If $(x,y)$ is in $P$, then $x < y$. Clearly, $y < x$ is impossible and the first axiom is satisfied. Also, inequality is transitive on numbers so the second axiom holds and this is a model.

1.6 Euclid’s Axiomatic Geometry

In this section we take a careful look at Euclid’s original axiomatic system. We observe some of its inadequacies in light of our modern “meta” understanding of such systems, and discuss the one axiom that has been the creative source of much of modern geometry – the Parallel Postulate.

1.6.1 Good research books for this question are books on the history of mathematics.

1.6.3 An explanation can be given based on a figure like the following:
Thus, \( \gcd(123, 36) = 3 \).

1.6.7 This exercise is a good starting off point for discussing the importance of definitions in mathematics. One possible definition for a circle is:

**Definition 1.1.** A circle with center \( O \) and radius length \( r \) is the set of points \( P \) on the sphere such that the distance along the great circle from \( O \) to \( P \) is \( r \).

Note that this definition is itself not entirely well-defined, as we have not specified what we mean by distance. Here, again, is a good opportunity to wrestle with the “best” definition of distance. For circles of any radius to exist, distance must be defined so that it grows without bound. Thus, one workable definition is for distance to be net cumulative arclength along a great circle as we move from a point \( O \) to a point \( P \).
SOLUTIONS TO EXERCISES IN CHAPTER 1

An angle \( ABC \) can be most easily defined as the Euclidean angle made by the tangent lines at \( B \) to the circles defining \( \overrightarrow{AB} \) and \( \overrightarrow{CB} \).

Then, Postulate 1 is satisfied as we can always construct a great circle passing through two points on the sphere. If the points are antipodal, we just use any great circle through those points. Otherwise, we simply intersect the sphere with the plane through the two points and the center of the sphere.

Postulate 2 is satisfied as we can always extend an arc of a great circle, although we may retrace the existing arc.

Postulate 3 is satisfied if we use the cumulative distance definition as discussed above.

Postulate 4 is automatically satisfied as angles are Euclidean angles.

Postulate 5 is not satisfied, as every pair of lines intersects. An easy proof of this is to observe that every line is uniquely defined by a plane through the origin. Two non-parallel planes will intersect in a line, and this line must intersect the sphere at two points.

1.6.9 This is true. Use a plane argument. Given a plane through the origin, we can always find an orthogonal plane. The angle these planes make will equal the angle of the curves they define on the sphere, as the spherical angles are defined by tangent lines to the sphere, and thus lie in the planes.

1.6.11 Yes. An example is the triangle that is defined in the first octant by intersecting the sphere with each of the three positive coordinate axes. This triangle has three right angles.

1.7 Project 2 - A Concrete Axiomatic System

After the last few sections dealing with abstract axiomatic systems, this lab is designed so that you can explore another geometric system through concrete manipulation of the points, lines, etc of that system. The idea here is to have you explore the environment first, then make some conjectures about what is similar and what is different in this system as compared to standard Euclidean geometry.
1.7.1 You should report the results of your experiments here. You do not yet have the tools to prove these results, but you should provide evidence that you have fully explored each idea.

For example, you could report that you tried to construct a rectangle, but were unsuccessful in doing so. You may discover that you construct a four-sided figure with three right angles, the fourth angle is always less than ninety degrees.

The sum of the angles in a triangle will be less than 180 degrees.

Euclid’s construction of an equilateral triangle is valid in hyperbolic geometry. Again, you should provide experimental evidence for this.

Finally, the perpendicular to a line through a point not on the line is a valid construction. Here, it is enough for you to experiment with the built-in perpendicular construction tool to create a new line that always stays perpendicular to a given line.
Chapter 2

Euclidean Geometry

In this chapter we start off with a very brief review of basic properties of angles, lines, and parallels.

Solutions to Exercises in Chapter 2

2.1 Angles, Lines, and Parallels

This section may be the least satisfying section in the chapter for you, since many theorems are referenced without proof. These results were (hopefully) covered in great detail in your high school geometry course and we will only briefly review them. A full and consistent development of the results in this section would entail a “filling in” of many days foundational work based on Hilbert’s axioms.

A significant number of the exercises in this section deal with parallel lines. This is for two reasons. First of all, historically there was a great effort to prove Euclid’s fifth Postulate by converting it into a logically equivalent statement that was hoped to be easier to prove. Thus, many of the exercises nicely echo this history. Secondly, parallels and the parallel postulate are at the heart of one of the greatest revolutions in math—the discovery of non-Euclidean...
CHAPTER 2. EUCLIDEAN GEOMETRY

geometry. This section foreshadows that development, which is covered in Chapters 7 and 8.

2.1.1 It has already been shown that \( \angle F BG \cong \angle DAB \). Also, by the vertical angle theorem (Theorem 2.3) we have \( \angle F BG \cong \angle EBA \) and thus, \( \angle DAB \cong \angle EBA \).

Now, \( \angle DAB \) and \( \angle CAB \) are supplementary, thus add to two right angles. Also, \( \angle CAB \) and \( \angle ABF \) are congruent by the first part of this exercise, as these angles are alternate interior angles. Thus, \( \angle DAB \) and \( \angle ABF \) add to two right angles.

2.1.3.a False, right angles are defined solely in terms of congruent angles.

2.1.3.b False, an angle is defined as just the two rays plus the vertex.

2.1.3.c True. This is part of the definition.

2.1.3.d False. The term “line” is undefined.

2.1.5 Proposition I-23 states that angles can be copied. Let \( A \) and \( B \) be points on \( l \) and \( n \) respectively and let \( m \) be the line through \( A \) and \( B \). If \( t = m \) we are done. Otherwise, let \( D \) be a point on \( n \) that is on the same side of \( n \) as \( l \). (Assuming the standard properties of betweenness) Then, \( \angle BAD \) is smaller than the angle at \( A \) formed by \( m \) and \( n \). By Theorem 2.9 we know that the interior angles at \( B \) and \( A \) sum to two right angles, so \( \angle CBA \) and \( \angle BAD \) sum to less than two right angles. By Euclid’s fifth postulate \( t \) and \( l \) must meet.

2.1.7 First, assume Playfair’s Postulate, and let lines \( l \) and \( n \) be parallel, with line \( t \) perpendicular to \( l \) at point \( A \). If \( t \) does not intersect \( m \) then, \( t \) and \( l \) are both parallel to \( m \), which contradicts Playfair. Thus, \( t \) intersects \( m \) and by Theorem 2.9 \( t \) is perpendicular at this intersection.

Now, assume that whenever a line is perpendicular to one of two parallel lines, it must be perpendicular to the other. Let \( l \) be a line and \( P \) a point not on \( l \). Suppose that \( m \) and \( n \) are both parallel to \( l \) at \( P \). Let \( t \) be a perpendicular from \( P \) to \( l \). Then, \( t \) is perpendicular to \( m \) and \( n \) at \( P \). By Theorem 2.4 it must be that \( m \) and \( n \) are coincident.
2.1.9 Assume Playfair and let lines \( m \) and \( n \) be parallel to line \( l \). If \( m \neq n \) and \( m \) and \( n \) intersect at \( P \), then we would have two different lines parallel to \( l \) through \( P \), contradicting Playfair. Thus, either \( m \) and \( n \) are parallel, or are the same line.

Conversely, assume that two lines parallel to the same line are equal or themselves parallel. Let \( l \) be a line and suppose \( m \) and \( n \) are parallel to \( l \) at a point \( P \) not on \( l \). Then, \( n \) and \( m \) must be equal, as they intersect at \( P \).

2.2 Congruent Triangles and Pasch’s Axiom

This section introduces many results concerning triangles and also discusses several axiomatic issues that arose from Euclid’s treatment of triangles.

2.2.1 Yes, it could pass through points \( A \) and \( B \) of \( \triangle ABC \). It does not contradict Pasch’s axiom, as the axiom stipulates that the line cannot pass through \( A \), \( B \), or \( C \).

2.2.3 No. Here is a counter-example.

\[\text{Figure 2.1:}\]

2.2.5 If \( A = C \) we are done. If \( A, B, \) and \( C \) are collinear, then \( B \) cannot be between \( A \) and \( C \), for then we would have two points...
CHAPTER 2. EUCLIDEAN GEOMETRY

of intersection for two lines. If $A$ is between $B$ and $C$, then $l$ cannot intersect $AC$. Likewise, $C$ cannot be between $A$ and $B$.

If the points are not collinear, suppose $A$ and $C$ are on opposite sides. Then $l$ would intersect all three sides of $\triangle ABC$, contradicting Pasch’s axiom.

2.2.7 Let $\angle ABC \cong \angle ACB$ in $\triangle ABC$. Let $\overrightarrow{AD}$ be the angle bisector of $\angle BAC$ meeting side $BC$ at $D$. Then, by AAS, $\triangle DBA$ and $\triangle DCA$ are congruent and $\overline{AB} \cong \overline{AC}$.

2.2.9 Suppose that two sides of a triangle are not congruent. Then, the angles opposite those sides cannot be congruent, as if they were, then by the previous exercise, the triangle would be isosceles.

Suppose in $\triangle ABC$ that $\overline{AC}$ is greater than $\overline{AB}$. On $\overline{AC}$ we can find a point $D$ between $A$ and $C$ such that $\overline{AD} \cong \overline{AB}$. Then, $\angle ADB$ is an exterior angle to $\triangle BDC$ and is thus greater than $\angle DCB$. But $\triangle ABD$ is isosceles and so $\angle ADB \cong \angle ABD$, and $\angle ABD$ is greater than $\angle DCB = \angle ACB$.

2.2.11 Let $\triangle ABC$ and $\triangle XYZ$ be two right triangles with right angles at $A$ and $X$, and suppose $\overline{BC} \cong \overline{YZ}$ and $\overline{AC} \cong \overline{XZ}$. Suppose $\overline{AB}$ is greater than $\overline{XY}$. Then, we can find a point $D$ between $A$ and $B$ such that $\overline{AD} \cong \overline{XY}$. By SAS $\triangle ADC \cong \triangle XYZ$. Now, $\angle BDC$ exterior to $\triangle ADC$ and thus must be greater than 90 degrees. But $\triangle CDB$ is isosceles, and thus $\angle DBC$ must also be greater than 90 degrees. This is impossible, as then $\triangle CDB$ would have angle sum greater than 180 degrees.
2.3 Project 3 - Special Points of a Triangle

You are encouraged to explore and experiment in this lab project. Are there any other sets of intersecting lines that one could construct for a given triangle? Are there interesting properties of constructed intersecting lines in other polygons?

2.3.1 \( \triangle DGB \) and \( \triangle DGA \) are congruent by SAS, as are \( \triangle EGB \) and \( \triangle EGC \). Thus, \( \triangle AG \cong \triangle BG \cong \triangle CG \). By SSS \( \triangle AFG \cong \triangle CFG \) and since the angles at \( F \) must add to 180 degrees, the angles at \( F \) must be congruent right angles.

2.3.3 The angle pairs in question are all pairs of an exterior angle and an interior angle on the same side for a line falling on two parallel lines. These are congruent by Theorem 2.9.

Since \( \angle DAB, \angle BAC, \) and \( \angle CAE \) sum to 180 degrees, and \( \angle BDA, \angle BAD, \) and \( \angle ABD \) sum to 180 then, using the congruences shown in the diagram, we get that \( \angle DBA \cong \angle BAC \). Likewise, \( \angle BAD \cong \angle ABC \). By ASA we get that \( \triangle ABC \cong \triangle BAD \). Similarly, \( \triangle ABC \cong \triangle CEA \) and \( \triangle ABC \cong \triangle FCB \).

2.3.5 Let \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) define an angle and let \( \overrightarrow{AD} \) be the bisector. Drop perpendiculars from \( D \) to \( \overline{AB} \) and \( \overline{AC} \), and assume these intersect at \( B \) and \( C \). Then, by AAS, \( \triangle ABD \) and \( \triangle ACD \) are congruent, and \( \overline{BD} \cong \overline{CD} \).

Conversely, suppose \( D \) is interior to \( \angle BAC \) with \( \overline{BD} \) perpendicular...
CHAPTER 2. EUCLIDEAN GEOMETRY

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ular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ perpendicular to $\overrightarrow{AC}$. Also, suppose that $\overrightarrow{BD} \cong \overrightarrow{CD}$. Then, by the Pythagorean Theorem $AB^2 + BD^2 = AD^2$ and $AC^2 + CD^2 = AD^2$. Thus, $\overline{AB} \cong \overline{AC}$ and by SSS $\triangle ABD \cong \triangle ACD$. This implies that $\angle BAD \cong \angle CAD$.

2.4.1 Mini-Project: Area in Euclidean Geometry

This section includes the first “mini-project” for the course. These projects are designed to be done in the classroom, in groups of three or four. Each group should elect a Recorder. The Recorder’s sole job is to outline the group’s solutions to exercises. The summary should not be a formal write-up of the project, but should give a brief synopsis of the group’s reasoning process.

The main goal for the mini-projects is to have discussion of geometric ideas. Through the group process, you can clarify your understanding of concepts, and help others better grasp abstract ways of thinking. There is no better way to conceptualize an idea than to have to explain it to another person.

In this mini-project, you are asked to grapple with the notion of “area”. The notion of area is not that simple or obvious. For example, what does it mean for two figures to have the same area?

2.4.1 Construct a diagonal and use the fact that alternate interior angles of a line falling on parallel lines are congruent to generate an ASA congruence for the two sub-triangles created in the parallelogram.

2.4.3 If the figure can be split into triangle pieces that can be separated into congruent pairs, then, since triangles are polygons, it can be split into congruent pairs of polygonal pieces.

On the other hand, it it can be split into congruent polygonal pieces, then we can split the polygon pieces into triangles, and we can use SAS repeatedly to generate congruent pairs of triangles.

2.4.5 Use Theorem 2.8 and Exercise 2.4.1.

Project Report

Hidden Assumptions? One hidden assumption is the notion that...
areas are additive. That is, if we have two figures that are not overlapping, then the area of the union is the sum of the separate areas.

2.4.2 Cevians and Area

2.4.7 Since a median is a cevian to a midpoint, then the fractions in the ratio product of Theorem 2.24 are all equal to 1.

2.4.9 Refer to the figure below. By the previous exercise we know that 1 + 2 + 3 = 4 + 5 + 6 (in terms of areas). Also, since 1 and 2 share the same base and height we have 3 = 4. Similarly, 1 = 5 and 5 = 6. Thus, 1 = 6.

Similarly, 2 + 3 + 4 = 1 + 5 + 6 will yield 4 = 5, and 3 + 4 + 5 = 1 + 5 + 5 yields 2 = 3. Thus, all 6 have the same area.

![Figure 2.3:](image)

2.5 Similar Triangles

As stated in the text, similarity is one of the most useful tools in the geometer’s toolkit. It can be used in the definition of the trigonometric functions and in proofs of theorems like the Pythagorean Theorem.

2.5.1 Since \( \overrightarrow{DE} \) cuts two sides of triangle at the midpoints, then by Theorem 2.27, this line must be parallel to the third side \( \overrightarrow{BC} \). Thus \( \angle ADE \cong \angle ABC \) and \( \angle AED \cong \angle ACB \). Since the angle at
is congruent to itself, we have by AAA that $\triangle ABC$ and $\triangle ADE$ are similar, with proportionality constant of $\frac{1}{2}$.

![Figure 2.4](image)

**2.5.3** Let $\triangle ABC$ and $\triangle DEF$ have the desired SSS similarity property. That is sides $AB$ and $DE$, sides $AC$ and $DF$, and sides $BC$ and $EF$ are proportional. We can assume that $AB$ is at least as large as $DE$. Let $G$ be a point on $AB$ such that $AG \cong DE$. Let $\overrightarrow{GH}$ be the parallel to $\overrightarrow{BC}$ through $G$. Then, $\overrightarrow{GH}$ must intersect $\overrightarrow{AC}$, as otherwise $\overrightarrow{AC}$ and $\overrightarrow{BC}$ would be parallel. By the properties of parallels, $\angle AGH \cong \angle ABC$ and $\angle AHG \cong \angle ACB$. Thus, $\triangle AGH$ and $\triangle ABC$ are similar.

Therefore, $\frac{AB}{AG} = \frac{AC}{AH}$. Equivalently, $\frac{AB}{DE} = \frac{AC}{DF}$. We are given that $\frac{AB}{DE} = \frac{AC}{DF}$. Thus, $AH \cong DF$.

Also, $\frac{AB}{AG} = \frac{BC}{GH}$ and $\frac{AB}{AG} = \frac{AB}{DE} = \frac{BC}{EF}$. Thus, $GH \cong EF$.

By SSS $\triangle AGH$ and $\triangle DEF$ are congruent, and thus $\triangle ABC$ and $\triangle DEF$ are similar.
SOLUTIONS TO EXERCISES IN CHAPTER 2

2.5.5 Any right triangle constructed so that one angle is congruent to $\angle A$ must have congruent third angles, and thus the constructed triangle must be similar to $\triangle ABC$. Since $\sin$ and $\cos$ are defined in terms of ratios of sides, then proportional sides will have the same ratio, and thus it does not matter what triangle one uses for the definition.

2.5.7 If the parallel to $\overrightarrow{AC}$ does not intersect $\overrightarrow{RP}$, then it would be parallel to this line, and since it is already parallel to $\overrightarrow{AC}$, then by exercise 2.1.15 $\overrightarrow{RP}$ and $\overrightarrow{AC}$ would be parallel, which is impossible.

By the properties of parallels, $\angle RAP \cong \angle RBS$ and $\angle RPA \cong \angle RSB$. Thus, by AAA $\triangle RBS$ and $\triangle RAP$ are similar. $\triangle PCQ$ and $\triangle SBQ$ are similar by AAA using an analogous argument for two of the angles and the vertical angles at $Q$.

Thus, $\frac{CP}{BS} = \frac{CQ}{BQ} = \frac{PQ}{QS}$, and $\frac{AP}{BS} = \frac{AR}{BR} = \frac{PR}{SR}$. So, $\frac{CP}{BS} \cdot \frac{BQ}{CQ} = \frac{AP}{BS} \cdot \frac{AR}{KB} = \frac{BS}{AP} \cdot \frac{AP}{BS} = 1$.

2.5.1 Mini-Project: Finding Heights

This mini-project is a very practical application of the notion of similarity. The mathematics in the first example for finding heights is not hard, but the interesting part is the data collection. You will need to determine how to get the most accurate measurements using the materials on hand.
CHAPTER 2. EUCLIDEAN GEOMETRY

The second method of finding height is a calculation using two similar triangles. The interesting part of this project is to see the connection between the mirror reflection and the calculation you made in part I.

You should work in small groups with a Recorder, but make sure the Recorder position gets shifted around from project to project.

2.6 Circle Geometry

This section is an introduction to the basic geometry of the circle. The properties of inscribed angles and tangents are the most important properties to focus on in this section.

2.6.1 Case I: $A$ is on the diameter through $OP$. Let $\alpha = m\angle PBO$ and $\beta = m\angle POB$. Then, $\beta = 180 - 2\alpha$. Also, $m\angle AOB = 180 - \beta = 2\alpha$.

Case II: $A$ and $B$ are on the same side of $\overset{\leftrightarrow}{PO}$. We can assume that $m\angle OPB > m\angle OPB$. Let $m\angle OPB = \alpha$ and $m\angle OPB = \beta$. Then, we can argue in a similar fashion to the proof of the Theorem using $\alpha - \beta$ instead of $\alpha + \beta$.

2.6.3 Consider $\angle AQO$ where $O$ is the center of the circle through $A$. This must be a right angle by Corollary 2.33. Similarly, $\angle BQO$ must be a right angle, where $O'$ is the center of the circle through $B$. Thus, $A$, $Q$, and $B$ are collinear.

2.6.5 Let $AB$ be the chord, $O$ the center, and $M$ the midpoint of $AB$. Then $\Delta AOM \cong \Delta BOM$ by SSS and the result follows.

2.6.7 Consider a triangle on the diagonal of the rectangle. This has a right angle, and thus we can construct the circle on this angle. Since the other triangle in the rectangle also has a right angle on the same side (the diameter of the circle) then it is also inscribed in the same circle.

2.6.9 Suppose they intersected at another point $P$. Then, $\Delta TBP$ and $\Delta TAP$ are both isosceles triangles. But, this would imply, by the previous exercise, that there is a triangle with two angles greater than a right angle, which is impossible.
SOLUTIONS TO EXERCISES IN CHAPTER 2

2.6.11 Let $P$ and $Q$ be points on the tangent, as shown. Then, $\angle BDT \cong \angle BTP$, as both are inscribed angles on the same arc. Likewise, $\angle ACT \cong \angle ATQ$. Since, $\angle BTP \cong \angle ATQ$ (vertical angles), then $\angle BDT \cong \angle ACT$ and the lines $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are parallel.

![Figure 2.6](image)

2.6.13 Suppose that the bisector did not pass through the center. Then, construct a segment from the center to the outside point. By the previous theorem, the line continued from this segment must bisect the angle made by the tangents. But, the bisector is unique and thus the original bisector must pass through the center.

2.7 Project 4 - Circle Inversion and Orthogonality

This section is crucial for the later development of the Poincaré model of non-Euclidean (hyperbolic) geometry. It is also has some of the most elegant mathematical results found in the course.

2.7.1 By Theorem 2.32, $\angle Q_2P_1P_2 \cong \angle Q_2Q_1P_2$. Thus, $\angle PP_1Q_2 \cong \angle PQ_1P_2$. Since triangles $\triangle PP_1Q_2$ and $\triangle PQ_1P_2$ share the angle at $P$, then they are similar. Thus, $\frac{PP_1}{PQ_1} = \frac{PQ_2}{PQ_2}$, or $(PP_1)(PP_2) = (PQ_1)(PQ_2)$.

2.7.3 By similar triangles $\frac{OP}{OT} = \frac{OT}{OP'}$. Since $OT = r$ the result follows.
Chapter 3

Analytic Geometry

This chapter is a very quick review of analytic geometry. In succeeding chapters, analytic methods will be utilized freely.

Solutions to Exercises in Chapter 3

3.2 Vector Geometry

3.2.1 If $A$ is on either of the axes, then so is $B$ and the distance result holds by the definition of coordinates. Otherwise, $A$ (and $B$) are not on either axis. Drop perpendiculars from $A$ and $B$ to the $x$ axis at $P$ and $Q$. By SAS similarity, $\triangle AOP$ and $\triangle BOQ$ are similar, and thus $\angle AOP \cong \angle BOQ$, which means that $A$ and $B$ are on the same line $\overrightarrow{AO}$, and the ratio of $BO$ to $AO$ is $k$.

3.2.3 The vector from $P$ to $Q$ is in the same direction (or opposite direction) as the vector $v$. Thus, since the vector from $P$ to $Q$ is $\overrightarrow{Q} - \overrightarrow{P}$, we have $\overrightarrow{Q} - \overrightarrow{P} = tv$, for some real number $t$. In coordinates we have $(x, y) - (a, b) = (tv_1, tv_2)$, or $(x, y) = (a, b) + t(v_1, v_2)$.

3.2.5 By exercise 3.2.3 the line through $A$ and $B$ can be represented by the set of points of the form $\overrightarrow{A} + t(\overrightarrow{B} - \overrightarrow{A})$. Then $M = \frac{1}{2}(\overrightarrow{A} + \overrightarrow{B}) = \overrightarrow{A} + \frac{1}{2}(\overrightarrow{B} - \overrightarrow{A})$ is on the line through $A$ and $B$, and is between $A$ and $B$. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then the
distance from \( A \) to \( M \) is \( \sqrt{\left( x_1 - x_2 \right)^2 + \left( y_1 - y_2 \right)^2} \), which is equal
to the distance from \( B \) to \( M \).

### 3.3 Project 5 - Bézier Curves

#### 3.3.1
The derivative to \( \vec{c}(t) \) is \( \vec{c}'(t) = 2\vec{B} - 2\vec{A} + 2t(\vec{C} - 2\vec{B} + \vec{A}) \). Then \( \vec{c}'(0) = 2\vec{B} - 2\vec{A} \), which is in the direction of \( \vec{B} - \vec{A} \) and \( \vec{c}'(1) = 2\vec{B} - 2\vec{A} + 2(\vec{C} - 2\vec{B} + \vec{A}) = 2\vec{C} - 2\vec{B} \), which is in the direction of \( \vec{C} - \vec{B} \).

#### 3.3.3
Similar computation to Exercise 3.3.1

### 3.4 Angles in Coordinate Geometry

#### 3.4.1
Let \( \vec{A} = (\cos(\alpha), \sin(\alpha)) \) and \( \vec{B} = (\cos(\beta), \sin(\beta)) \). Then, from Theorem 3.11 we have \( \cos(\alpha - \beta) = \vec{A} \circ \vec{B} \), since \( \vec{A} \) and \( \vec{B} \) are unit length vectors. The result follows immediately.

##### 3.4.3
By exercise 3.4.1,

\[
\cos\left( \frac{\pi}{2} - (\alpha + \beta) \right) = \cos\left( \frac{\pi}{2} \right) \cos(\alpha + \beta) + \sin\left( \frac{\pi}{2} \right) \sin(\alpha + \beta) \\
= \sin(\alpha + \beta).
\]

Then, use the formula from Exercise 3.4.2 with the term inside cosine being \( \left( \frac{\pi}{2} - \alpha \right) + (-\beta) \).

### 3.5 The Complex Plane

#### 3.5.1

\[
e^{i\theta}e^{i\phi} = (\cos(\theta) + i \sin(\theta))(\cos(\phi) + i \sin(\phi)) \\
= (\cos(\theta) \cos(\phi) - \sin(\theta) \sin(\phi)) + i(\cos(\theta) \sin(\phi) + \sin(\theta) \cos(\phi)) \\
= \cos(\theta + \phi) + i \sin(\theta + \phi) \\
= e^{i(\theta + \phi)}
\]

#### 3.5.3
Let \( z = e^{i\theta} \) and \( w = e^{i\phi} \) and use Exercise 3.4.1.
3.5.5 The rationalized complex numbers have the form \( i \frac{-1}{2} \), and \( \frac{1}{10} - i \frac{1}{5} \).

3.5.7 The line through \( N, P', \) and \( P \) can be expressed as \( t(P' - N) = (0, 0, 1) + t(X, Y, Z - 1) = (tX, tY, 1 + t(Z - 1)). \) As this is a point in the \( x - y \) plane, we have that. 1 + t(Z - 1) = 0, or \( t = \frac{1}{1-Z} \). Thus, \( \pi(P') = t(X, Y) = \frac{1}{1-Z} (X, Y) \).

3.5.9 Let \( z = (x, y) \) be a point in the complex plane. Then, \( (X, Y, Z) = \left( \frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right) \) will get mapped to \( z \) from the work done in exercises 3.4.5 and 3.4.6.

3.5.11 Since \( |z - z_0| = |z - z_0| \), the function \( f(z) = z \) has the local scale-preserving property. Consider two curves \( c_1 \) and \( c_2 \) intersecting at \( z_0 \), parameterized so that \( c_1(0) = c_2(0) = z_0 \). Then, the angle between their tangents is the argument of \( c_1'(0) \) minus the argument of \( c_2'(0) \). Under conjugation, the arguments become negative, and thus, the difference in the angles between the conjugate curves becomes negative.

3.6 Birkhoff’s Axiomatic System for Analytic Geometry

3.6.1 First, if \( A \) is associated to \( x_A = t_A \sqrt{dx^2 + dy^2} \), where \( A = (x, y) = (x_0, y_0) + t_A(dx, dy) \), and \( B \) is associated to \( x_B \) in a similar fashion, then \( |x_A - x_B| = |t_A - t_B| \sqrt{dx^2 + dy^2} \). On the other hand,

\[
d(A, B) = \sqrt{(t_A dx - t_B dx)^2 + (t_A dy - t_B dy)^2} = \sqrt{dx^2 + dy^2 |t_A - t_B|^2}.
\]

3.6.3 Given a point \( O \) as the vertex of the angle, set \( O \) as the origin of the coordinate system. Then, identify a ray \( \overrightarrow{OA} \) associated to the angle \( \theta \), with \( A = (x, y) \). Let \( a = ||A|| = \sqrt{x^2 + y^2} \). Then, \( \sin^2(\theta) + \cos^2(\theta) = \left( \frac{x}{a} \right)^2 + \left( \frac{y}{a} \right)^2 = \frac{x^2 + y^2}{a^2} = 1 \).

3.6.5 Discussion question. One idea is that analytic geometry allows one to study geometric figures by the equations that define them. Thus, geometry can be reduced to the arithmetic (algebra) of equations.
Chapter 4

Constructions

In this chapter we cover some of the basic Euclidean constructions and also have a lot of fun with lab projects. The origami project should be especially interesting, as it is an axiomatic system with which you can physically interact and explore.

The third section on constructibility may be a bit heavy and abstract, but the relationship between geometric constructibility and algebra is a fascinating one, especially if you have had some exposure to abstract algebra. Also, any mathematically literate person should know what the three classical construction problems are, and how the pursuit of solutions to these problems has had a profound influence on the development of modern mathematics.

Solutions to Exercises in Chapter 4

4.1 Euclidean Constructions

4.1.1 Use SSS triangle congruence on $\Delta ABF$ and $\Delta DGH$.

4.1.3 Use the SSS triangle congruence theorem on $\Delta ADE$ and $\Delta ABE$ to show that $\angle EAB \cong \angle BAE$.

4.1.5 Use the fact that both circles have the same radius.

4.1.7 Let the given line be $l$ and let $P$ be the point not on $l$.
Construct the perpendicular \( m \) to \( l \) through \( P \). At a point \( Q \) on \( m \), but not on \( l \), construct the perpendicular \( n \) to \( m \). Theorem 2.8 implies that \( l \) and \( n \) are parallel.

4.1.9 On \( \vec{BA} \) construct \( A' \) such that \( BA' = a \). On \( \vec{BC} \) construct \( C' \) such that \( BC' = b \). Then, SAS congruence gives \( \triangle AB'C' \) congruent to any other triangle with the specified data.

4.2 Project 6 - Euclidean Eggs

4.2.1 The tangent to one of the circles will meet \( \vec{AB} \) at \( C \) at right angles by Theorem 2.36. The tangent to the other circle will also meet \( \vec{AB} \) at \( C \) at a right angle. Since the perpendicular to \( \vec{AB} \) at \( C \) is unique, the tangents coincide.

4.2.3 The construction steps are implied by the figure.

4.3 Constructibility

4.3.1 Just compute the formula for the intersection.

4.3.3 Reverse the roles of the product construction.

4.3.5 For \( \sqrt{3} \), use a right triangle with hypotenuse 2 and one side 1. For \( \sqrt{5} \), use a right triangle with sides of length 1 and 2.

4.3.7 Consider \( \frac{a}{n} \). This is less than \( a \).

4.3.9 If a circle of radius \( r \) and center \((x, y)\) has \( x \) not constructible, then \((x, y + r)\) and \((x, y - r)\) are non-constructible on the circle. We can use the same reasoning if \( y \) is not constructible. If the center is constructible, then the previous exercise gives at least two non-constructible points for a circle of radius \( r \) whose center is at the origin. Add \((x, y)\) to these two points to get two non-constructible points on the original circle.

4.4 Mini-Project: Origami Construction

For this project, one will need a good supply of square paper. Commercial origami paper is quite expensive. Equally as good paper can
be made by taking notepads and cutting them into squares using a paper-cutter. (Cutting works best a few sheets at a time)

4.4.1 Given $\overline{AB}$, we can fold $A$ onto $B$ by axiom O2. Let $D$ be the fold line of reflection created, and let $l$ intersect $AB$ at $C$. Then, since the fold preserves length, we have that $AC = CB$, and $\angle ACE \cong \angle ECB$, as shown in Fig. 4.1. The result follows.

Figure 4.1:

4.4.3 Since the reflection fold across $t$ preserves length, we have $PR = P'R$. Also, the distance from a point to a line is measured along the perpendicular from the point to the line. Thus, the distance from $R$ to $l$ is equal to $P'R$. Thus, the distance from $R$ to $l$ equals the distance from $R$ to $l$ and $R$ is on the parabola with focus $P$ and directrix $l$.

An interesting result related to this construction would be to show that $t$ is tangent to the parabola at $R$. One proof is as follows:

Suppose $t$ intersected at another point $R'$ on the parabola. Then, by definition, $R'$ must have been constructed in the same way that $R$ was, so there must be a folding (reflection) across $t$ taking $P$ to some point $P''$ on $l$ such that $\overleftrightarrow{P''R'}$ is perpendicular to $l$ at $P''$, and intersects $t$ at $R'$. Then, by a triangle argument, we can show that $\overleftrightarrow{PP'}$ and $\overleftrightarrow{PP''}$ must both be perpendicular to $t$ at $R$ and $R'$. Since perpendiculars are unique, we must have that $R = R'$.

(To show, for example, that $\overleftrightarrow{PP'}$ is perpendicular to $t$ at $R$, we...
can easily show that $\triangle PQR \cong \triangle P'QR$ by using the angle- and distance-preserving properties of reflections, and then use a second congruent triangle argument to show that $\overrightarrow{PP'}$ crosses $t$ at right angles.)
Chapter 5

Transformational Geometry

In this chapter we make great use of functional notation and somewhat abstract notions such as $1 - 1$ and onto, inverses, composition, etc. You may wonder how such computations are related to geometry, but that is the very essence of the chapter—that we can understand and investigate geometric ideas with more than one set of mathematical techniques.

With that in mind, we will make use of synthetic geometric techniques where they are most elegant and can aid intuition, and at other times we will rely on analytical techniques.

Solutions to Exercises in Chapter 5

5.1 Euclidean Isometries

5.1.1 Define the function $f^{-1}$ by $f^{-1}(y) = x$ if and only if $f(x) = y$.

Then, $f^{-1}$ is well-defined, as suppose $f(x_1) = f(x_2) = y$. Then, since $f$ is $1 - 1$ we have that $x_1 = x_2$. Since $f$ is onto, we have that for every $y$ in $S$ there is an $x$ such that $f(x) = y$. Thus, $f^{-1}$ is defined on all of $S$. Finally, $f^{-1}(f(x)) = f^{-1}(y) = x$ and
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\[ f(f^{-1}(y)) = f(x) = y. \] So, \( f \circ f^{-1} = f^{-1} \circ f = \text{id}_S. \)

Suppose \( g \) was another function on \( S \) such that \( f \circ g = g \circ f = \text{id}_S. \) Then, \( g \circ f \circ f^{-1} = f^{-1}, \) or \( g = f^{-1}. \)

5.1.3 Since \( g^{-1} \circ f \circ g = g^{-1} \circ g = \text{id} \) and \( f \circ g \circ g^{-1} \circ f^{-1} = f \circ f^{-1} = \text{id}, \) then \( g^{-1} \circ f^{-1} = (f \circ g)^{-1}. \)

5.1.5 Let \( T \) be an isometry and let \( c \) be a circle centered at \( O \) of radius \( r = \overline{OA}. \) Let \( O' = T(O) \) and \( A' = T(A). \) Let \( P \) be any point on \( c. \) Then, \( O'T(P) = T(O)T(P) = OP = r. \) Thus, the image of \( P \) under \( T \) is contained in the circle centered at \( O' \) of radius \( r. \) Let \( P' \) be any other point on the circle centered at \( O' \) of radius \( r. \) Then, \( OT^{-1}(P') = T^{-1}(O')T^{-1}(P') = O'P' = r. \) Thus, \( T^{-1}(P') \) is a point on \( c \) and every such point \( P' \) is the image of a point on \( c, \) under the map \( T. \)

5.1.7 Label the vertices of the triangle \( A, B, \) and \( C. \) Then, consider vertex \( A. \) Under an isometry, consider the actual position of \( A \) in the plane. After applying the isometry, \( A \) might remain or be replaced by one of the other two vertices. Thus, there are three possibilities for the position occupied by \( A. \) Once that vertex has been identified, consider position \( B. \) There are now just two remaining vertices to be placed in this position. Thus, there are maximum of 6 isometries. We can find 6 by considering the three basic rotations by 0, 120, and 240 degrees, and the three reflections about perpendicular bisectors of the sides.

5.1.9 First, we show that \( T \) is a transformation. To show it is 1 – 1, suppose \( T(x, y) = T(x', y'). \) Then, \( kx + a = kx' + a \) and \( ky + b = ky' + b. \) So, \( x = x' \) and \( y = y'. \)

To show it is onto, let \( (x', y') \) be a point. Then, \( T(x' - a, y' - b) = (x', y'). \)

\( T \) is not, in general, an isometry, since if \( A = (x, y) \) and \( B = (x', y') \) then \( T(A)T(B) = kAB. \)

5.1.11 Let \( ABC \) be a triangle and let \( A'B'C' \) be its image under \( T. \) By the previous exercise, these two triangles are similar. Thus, there is a \( k > 0 \) such that \( A'B' = kAB, B'C' = kBC, \) and \( A'C' = kAC. \) Let \( D \) be any other point not on \( \overrightarrow{AB}. \) Then, using triangles...
ABD and \(A'B'D'\) we get that \(A'D' = kAD\).

Now, let \(\overrightarrow{DE}\) be any segment with \(D\) not on \(\overrightarrow{AB}\). Then, using triangles \(ADE\) and \(A'D'E'\) we get \(D'E' = kDE\), since we know that \(A'D' = kAD\).

Finally, let \(\overrightarrow{EF}\) be a segment entirely on \(\overrightarrow{AB}\), and let \(D\) be a point off \(\overrightarrow{AB}\). Then, using triangles \(DEF\) and \(D'E'F'\) we get \(E'F' = kEF\), since we know that \(D'E' = kDE\).

Thus, in all cases, we get that \(T(A)T(B) = kAB\).

### 5.2.1 Mini-Project: Isometries Through Reflection

In this mini-project, you will be led through a guided discovery of the amazing fact that, given any two congruent triangles, one can find a sequence of at most three reflections taking one triangle to the other.

**5.2.1** First of all, suppose that \(C\) and \(R\) are on the same side of \(\overrightarrow{AB}\). Then, since there is a unique angle with side \(\overrightarrow{AB}\) and measure equal to the measure of \(\angle BAC\), then \(R\) must lie on \(\overrightarrow{AC}\). Likewise, \(R\) must lie on \(\overrightarrow{BC}\). But, the only point common to these two rays is \(C\). Thus, \(R = C\).

If \(C\) and \(R\) are on different sides of \(\overrightarrow{AB}\), then drop a perpendicular from \(C\) to \(\overrightarrow{AB}\), intersecting at \(P\). By SAS, \(\triangle P\overrightarrow{AC}\) and \(\triangle P\overrightarrow{AR}\) are congruent, and thus \(\angle APR\) must be a right angle, and \(R\) is the reflection of \(C\) across \(\overrightarrow{AB}\).

**5.2.3** If two triangles (\(\triangle ABC\) and \(\triangle PQR\)) share no point in common, then by Theorem 5.6 there is a reflection mapping \(A\) to \(C\), and by the previous exercise, we would need at most two more reflections to map \(\triangle r(A)r(B)r(C)\) to \(\triangle PQR\).

### 5.2.2 Reflections

**5.2.5** Many example from nature have bilateral symmetry.

**5.2.7** Let \(G\) be the midpoint of \(\overrightarrow{AB}\). Then \(\triangle AED \cong \triangle BCD\) by SAS and \(\triangle AGD \cong \triangle BDG\) by SSS. Thus, \(\overrightarrow{DG}\) is the perpendicular...
bisection of $\overline{AB}$, and reflection across $\overrightarrow{DG}$ takes $A$ to $B$. Also, $\overrightarrow{DG}$ must bisect the angle at $D$ and by the previous exercise the bisection is a line of reflection. This proof would be easily extendable to regular $n$-gons, for $n$ odd, by using repeated triangle congruences to show the perpendicular bisector is the angle bisector of the opposite vertex.

![Figure 5.1:](image)

5.2.9 Suppose that a line of symmetry $l$ for parallelogram $ABCD$ is parallel to side $\overline{AB}$. Then, clearly reflection across $l$ cannot map $A$ to $B$, as this would imply that $l$ is the perpendicular bisector of $\overline{AB}$.

If reflection mapped $A$ to $C$, then $l$ would be the perpendicular bisector of a diagonal of the parallelogram. But, since $l$ is parallel to $\overline{AB}$, this would imply that the diagonal must be perpendicular to $\overline{AB}$ as well. A similar argument can be used to show that the other diagonal ($\overline{BD}$) must also be perpendicular to $\overline{AB}$. If this were the case, one of the triangles formed by the diagonals would have angle sum greater than 180 degrees, which is impossible.

Thus, reflection across $l$ must map $A$ to $D$, and $l$ must be the perpendicular bisector of $\overline{AD}$. Clearly, using the property of parallels, we get that the angles at $A$ and $D$ in the parallelogram are right angles.
5.2.11 Let $r$ be a reflection across $\overrightarrow{AB}$ and let $C$ be a point not on $\overrightarrow{AB}$. Then, $r(C)$ is the unique point on the perpendicular dropped to $\overrightarrow{AB}$ at a point $P$ on this line such that $CP = r(C)P$ with $r(C) \neq C$. Now, $r(r(C))$ is the unique point on this same perpendicular such that $r(C)P = r(r(C))P$, with $r(r(C)) \neq r(C)$. But since $r(C)P = CP$ and $C \neq r(C)$, then $r(r(C)) = C$. But, then $r \circ r$ fixes three non-collinear points $A$, $B$, and $C$, and so must be the identity.

5.2.13 Let $A$ and $B$ be distinct points on $l$. Then, $r_{m} \circ r_{l}(r_{m}(A)) = r_{m}(r_{l}(A)) = r_{m}(A)$ and likewise, $r_{m} \circ r_{l} \circ r_{m}(r_{m}(B)) = r_{m}(B)$. Thus, the line $l'$ through $r_{m}(A)$ and $r_{m}(B)$ is fixed by $r_{m}$, $r_{l}$, and this triple composition must be equivalent to reflection across $l'$.

5.2.15 Drop a perpendicular from $O$ to the line intersecting at $Q$. By SAS we get the length from $O$ to $P$ is the same as the length from $O'$ to $P$. Thus, to minimize the total length to $V$ we just minimize the length from $O'$ to $P$ to $V$. But, the shortest path will be a straight line, so $P$ must be located so that it is on the line through $O'$ and $V$. Using congruent triangles and vertical angles, we see that the shortest path occurs when the two angles made at $P$ are congruent.

5.3 Translations

5.3.1 There are few examples in nature that have perfect translational symmetry. One example might be the atoms in a crystal atomic lattice. But there are some partial examples, like the legs of a millipede.

5.3.3 Since $(r_{2} \circ r_{1}) \circ (r_{1} \circ r_{2}) = id$, and $(r_{1} \circ r_{2}) \circ (r_{2} \circ r_{1}) = id$, then $r_{2} \circ r_{1}$ is the inverse of $r_{1} \circ r_{2}$. Also, if $T$ has translation vector $v$, then $T(x, y) = (x, y) + v$. Let $S$ be the translation defined by $S(x, y) = (x, y) - v$. Then, $S \circ T(x, y) = ((x, y) + v) - v = (x, y)$ and $T \circ S((x, y) - v) + v = (x, y)$. Thus, $S$ is the inverse to $T$.

5.3.5 Let $T_{1}$ have translation vector $v_{1}$ and $T_{2}$ have translation
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vector $v_2$. Then, $T_1 \circ T_2(x, y) = T_1((x, y) + v_2) = (x, y) + (v_2 + v_1)$, which is the same as $T_1 \circ T_2(x, y)$.

5.3.7 Let $(x, K)$ be a point on the line $y = K$. If $T$ is a translation with translation vector $v = (0, -K)$, then, by exercise 5.3.3, $T^{-1}$ has translation vector of $-v = (0, K)$. Thus, $T^{-1} \circ r_x \circ T(x, K)$ is $T^{-1} \circ r_x(x, 0) = T^{-1}(x, 0) = (x, K)$. So, $T^{-1} \circ r_x \circ T$ fixes the line $y = K$ and so must be the reflection across this line. The coordinate equation for $r$ is given by $T^{-1} \circ r_x \circ T(x, y) = T^{-1} \circ r_x(x, y - K) + T^{-1}(x, -y + K) = (x, -y + 2K)$. So, $r(x, y) = (x, -y + 2K)$.

5.3.9 Let $T$ be a translation with (non-zero) translation vector parallel to a line $l$. Let $m$ be perpendicular to $l$ at point $P$. Let $n$ be the perpendicular bisector of $PT(P)$, intersecting $PT(P)$ at point $Q$. Then, $r_n$, reflection about $n$ maps $P$ to $T(P)$. Consider $r_n \circ T$. We have $r_n \circ T(P) = P$. Let $R \neq P$ be another point on $m$. Then, $PRT(R)T(P)$ is a parallelogram, and thus $\angle PRT(R)$ and $\angle RT(R)T(P)$ are right angles. Let $S$ be the point where $n$ intersects $RT(R)$. Then, $\angle RSQ$ is also a right angle. Also, by congruent triangle argument, we have $RS \cong ST(R)$, and so $n$ is the perpendicular bisector of $RT(R)$ and $r_n \circ T(R) = R$. Since $r_n \circ T$ fixes two points on $m$ we have $r_n \circ T = r_m$, or $T = r_n \circ r_m$.

Figure 5.2:
5.4 Rotations

5.4.1 First,

\[ T^{-1} \circ \text{Rot}_\phi \circ T(C) = T^{-1} \circ \text{Rot}_\phi \circ T(x, y) = T^{-1} \circ \text{Rot}_\phi (0, 0) = T^{-1}(0, 0) = (x, y) = C \]

Suppose \( T^{-1} \circ \text{Rot}_\phi \circ T \) fixed another point \( P \). Then, \( \text{Rot}_\phi \) \( T(P) = T(P) \), which implies that \( T(P) = (0, 0) \), or \( P = T^{-1}(0, 0) \) \( (x, y) = C \). Thus, \( T^{-1} \circ \text{Rot}_\phi \circ T \) must be a rotation. What is the angle for this rotation? Consider a line \( l \) through \( C \) that is parallel to the \( x \)-axis. Then, \( T \) will map \( l \) to the \( x \)-axis and \( \text{Rot}_\phi \) will map the \( x \)-axis to a line \( m \) making an angle of \( \phi \) with the \( x \)-axis. Then, \( T^{-1} \) will preserve this angle, mapping \( m \) to a line making an angle of \( \phi \) with \( l \). Thus, the rotation angle for \( T^{-1} \circ \text{Rot}_\phi \circ T \) is \( \phi \).

5.4.3 A book on flowers or diatoms (algae) would be a good place to start.

5.4.5 By the preceding exercise, the invariant line must pass through the center of rotation. Let \( A \) be a point on the invariant line. Then, \( R_O(A) \) lies on \( \overrightarrow{OA} \) and \( \overline{OA} \cong \overline{OR_O(A)} \). Either \( A \) and \( R_O(A) \) are on the same side of \( O \) or are on opposite sides. If they are on the same side, then \( A = R_O(A) \), and the rotation is the identity which is ruled out. If they are on opposite sides, then the rotation is 180 degrees. If the rotation is 180 degrees, then for every point \( A \neq O \) we have that \( A \), \( O \), and \( R_O(A) \) are collinear, which means that the line \( \overrightarrow{OA} \) is invariant.

5.4.7 Let \( R = r_l \circ r_m \) be a rotation about the point \( P \) where \( l \) and \( m \) intersect. Then, since \((r_l \circ r_m) \circ (r_m \circ r_l) = id \) and \((r_m \circ r_l) \circ (r_l \circ r_m) = id \), then \( R^{-1} = r_m \circ r_l \), and the angle of rotation is the same, but in reverse direction, as the angle is twice the angle between the lines of reflection.
5.4.9 Consider $R^{-1} \circ R'$. This map fixes $O$ and $A$ and thus fixes $\overrightarrow{OA}$. So, either $R^{-1} \circ R'$ is a reflection or it is the identity. Since the composition of two rotations about a common point is again a rotation (by the preceding exercise), then $R^{-1} \circ R' = id$ and the result follows.

5.4.11 The hint is over-kill. $H$ is clearly a rotation, by the definition of rotations. The angle of rotation is twice the angle made by the lines of reflection, or twice a right angle, or 180.

5.4.13 Note that $T \circ H_A \circ T^{-1}$ maps $T(A)$ back to itself. This map fixes any other point $P$, then $H_A \circ T^{-1}(P) = T^{-1}(P)$, and so $T^{-1}(P) = A$ or $P = T(A)$. Thus, $T \circ H_A \circ T^{-1}$ is a rotation about $T(A)$. Then, any line through $T(A)$ will get mapped to a line through $A$ by $T^{-1}$. Then $H_A$ will map this new line to itself, and $T$ will map this half-turned line back to the original line. Thus, by exercise 5.4.5, $T \circ H_A \circ T^{-1}$ is a half-turn about $T(A)$.

5.5 Project 7 - Quilts and Transformations

This project is another great opportunity for the future teachers in the class to develop similar projects for use in their own teaching. One idea to incorporate into a high school version of the project is to bring into the class the cultural and historical aspects of quilting.

5.5.1 In your Project Report give a report of how you did the construction.

5.5.3 For bilateral symmetry, any reflection line must pass through the center of the quilt pattern. The only patterns which have such symmetry are: 25-Patch Star (horizontal, vertical, 45 degree, and $-45$ degree lines of symmetry) and Flower Basket (45 degree line of symmetry).

Star Puzzle, Dutch Man’s Puzzle, and 25-Patch Star all have rotational symmetry of 90 (and thus 180 and 270) degrees.

Thus, 25-Patch Star is the only pattern with both rotational and bilateral symmetry.


SOLUTIONS TO EXERCISES IN CHAPTER 5

5.6 Glide Reflections

5.6.1 As with translations, it will be hard to find a perfect example of a glide symmetry in nature. But, there are many plants whose branches alternate in a glide fashion.

5.6.3 Suppose \( m \) is invariant. Then, the glide reflection can be written as \( G = T_{AB} \circ r_l = r_l \circ T_{AB} \). If \( G(G(m)) = m \), then
\[
(T_{AB} \circ r_l) \circ (r_l \circ T_{AB})(m) = T_{2AB}(m) = m.
\]
So, \( m \) must be parallel or equal to \( l \), if it is invariant under \( T_{2AB} \). Suppose \( m \) is parallel to \( l \). Then, \( T_{AB}(m) = m \). So, \( G(m) = r_l \circ T_{AB}(m) = r_l(m) \). But, the reflection of a line \( m \) that is parallel to \( l \) cannot be equal to \( m \). Thus, the only line invariant under the glide reflection is \( l \) itself.

5.6.5 The glide reflection can be written as \( G = T_{AB} \circ r_l = r_l \circ T_{AB} \). So, \( G \circ G = (T_{AB} \circ r_l) \circ (r_l \circ T_{AB}) = T_{2AB} \).

5.6.7 The set does not include the identity element.

5.6.9 The identity (rotation angle of 0) is in the set. The composition of two rotations about the same point is again a rotation by exercise 5.4.8. The inverse to a rotation is another rotation about the same point by exercise 5.4.7. Since rotations are functions, associativity is automatic.

5.6.11 A discussion and diagram would suffice for this exercise.

5.6.13 By using the result in Exercise 5.2.14 repeatedly, we can reduce any even (non-identity) isometry to the product of two reflections. Also, the identity can be written as the product of two reflections, the product of a reflection with itself. An odd isometry can be reduced to the product of three or one reflections. Since rotations and translations cannot be equivalent transformations to reflections and glide reflections, then an isometry cannot be both even and odd.

5.7 Structure and Representation of Isometries

This section is a somewhat abstract digression into ways of representing transformations and of understanding their structure as algebraic elements of a group. An important theme of the section...
CHAPTER 5. TRANSFORMATIONAL GEOMETRY

the usefulness of the matrix form of an isometry, both from a theoretical viewpoint (classification), as well as a practical viewpoint (animation in computer graphics).

Matrix methods (and thus transformations) are used heavily in the field of computer animation. There are many excellent textbooks in computer graphics that one could use as reference for this purpose. For example, the book by F.S. Hill listed in the bibliography of this text is a very accessible introduction to the subject.

5.7.1 Let $G_1 = T_{v_1} \circ r_{l_1}$ and $G_2 = T_{v_2} \circ r_{l_2}$ be two glide reflections. If $G_1 \circ G_2$ is a translation, say $T_v$, then, since $G_1 \circ G_2 = T_v \circ (T_{l_1} \circ r_{l_1}) \circ (r_{l_2} \circ T_{l_2})$, then $T_{v-v_1-v_2} = r_{l_1} \circ r_{l_2}$ and thus $l_1 || l_2$.

On the other hand, if the lines are parallel, then $G_1 \circ G_2 = (T_{v_1} \circ r_{l_1}) \circ (r_{l_2} \circ T_{v_2}) = T_{v_1} \circ T_v \circ T_{v_2}$, for some vector $v$.

If the lines intersect, then the composition of $r_{l_1}$ with $r_{l_2}$ will be a rotation, say $R$, and $G_1 \circ G_2 = (T_{v_1} \circ r_{l_1}) \circ (r_{l_2} \circ T_{v_2})$. This last composition yields a rotation, by Theorem 5.20.

5.7.3 First, $f \circ r_m \circ f^{-1}(f(m)) = f(m)$, so $f(m)$ is a fixed line for $f \circ r_m \circ f^{-1}$. Also, $(f \circ r_m \circ f^{-1})^2 = f \circ r_m \circ f^{-1} \circ f \circ r_m \circ f^{-1} = id$. Thus, $f \circ r_m \circ f^{-1}$, which must be a reflection or glide reflection from looking at Table 5.3, is a reflection. Since it fixes $f(m)$ it must be a reflection across $f(m)$.

5.7.5 Using the previous exercises we have $f \circ r_m \circ T_{AB} \circ f^{-1} = f \circ r_m \circ f^{-1} \circ f \circ T_{AB} \circ f^{-1} = r_m \circ T_f(A) \circ f(B)$.

5.7.7 Rotation of $(x, y)$ by an angle $\phi$ yields $(x \cos(\phi) - y \sin(\phi), y \cos(\phi))$. Multiplying $x + iy$ by $\cos(\phi) + i \sin(\phi)$ yields the same result. Translation by $v = (v_1, v_2)$ yields $(x + v_1, y + v_2)$. Adding $v_1 + iv_2$ to $x + iy$ yields the same result. Finally, reflection across $y = \frac{x - y}{2}$ is given by $r_x(x, y) = (x, -y)$. Complex conjugation sends $x + iy$ to $x - iy$. Clearly, this has the same effect.

5.7.9 $T_v \circ R_{\beta}(z) = (e^{i\beta}z) + v$. To find the fixed point set $(e^{i\beta}z) + v = z$ and solve for $z$. 

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5.8 Project 8 - Constructing Compositions

The purpose of this lab is to make concrete the somewhat abstract notion of composition of isometries. In particular, by carrying out the constructions of the lab, you will see how the conditions on compositions of rotations found in Table 5.3 arise naturally.

If you have difficulty getting started with the first proof, think about how we can write a rotation as the composition of two reflections through the center of rotation. Note that the choice of reflection lines is not important – one can choose any two lines as long as they make the right angle, namely half the desired rotation angle.

5.8.1 A rotation can be expressed as the composition of two reflections about lines through the center of rotation, as long as the reflection lines make an angle of half the reflection angle. Since \( n \) and \( m \) are bisectors of the rotation angles, then, \( R_{A, \angle EAB} = r_n \circ r_{\overrightarrow{AE}} \) and \( R_{B, \angle ABE} = r_{\overrightarrow{AB}} \circ r_m \), taking into account the orientation of the rotation angles.

5.8.3 The rotation angle \( \gamma \) is twice the angle at \( O \) in \( \triangle AOB \). This angle is \( \angle BOA \). (Note - positively oriented) Then, taking care to measure orientation correctly, we have

\[
\gamma = 2(180 - (\angle BAO + \angle OBA))
= 360 + (2\angle OAB + 2\angle ABO)
= 360 + (\angle EAB + \angle ABE)
\]

Thus, \( \gamma = (\angle EAB + \angle ABE) \pmod{360} \).
Chapter 6

Symmetry

This chapter is quite algebraic in nature—focusing on the different discrete symmetry groups that arise for frieze patterns and wallpaper patterns.

Solutions to Exercises in Chapter 6

6.1 Finite Plane Symmetry Groups

6.1.1 Flowers and diatoms make good examples.

6.1.2 The symmetry group is the dihedral group of order 4. (4 rotations generated by a rotation of 90 degrees, and reflections generated by a reflection across a perpendicular bisector of a side) This gives 8 symmetries. There are no more, since if we label the vertices and fix a position for a vertex to occupy, we have 4 choices for the vertex to be placed in that position and only two choices for the rest of the vertices. Thus, a maximum of eight symmetries possible.

6.1.3 The dihedral group of order 5. (5 rotations generated by a rotation of 72 degrees, and reflections generated by a reflection across a perpendicular bisector of a side) This gives 10 symmetries. There are no more, since if we label the vertices and fix a position
for a vertex to occupy, we have 5 choices for the vertex to be placed in that position and only two choices for the rest of the vertices. Thus, a maximum of ten symmetries possible.

6.1.5 Using an argument like that used in exercises 6.1.2 and 6.1.3, we know there are at most $2n$ symmetries. Also, by the work done in section 5.4 we know there are $n$ rotations, generated by a rotation of $\frac{360}{n}$, that will be symmetries. Let $r$ be a reflection across a perpendicular bisector of a side. This will be a reflection, as will all $n$ compositions of this reflection with the $n$ rotations. This gives $2n$ different symmetries.

6.1.7 The number of symmetries is $2n$. The only symmetries that fix a side are the identity and a reflection across the perpendicular bisector of that side. The side can move to $n$ different sides. Thus, the stated product is $2n$ as claimed.

6.2 Frieze Groups

6.2.1 Since $\gamma^2 = \tau$, then $< \tau, \gamma, H >$ is contained in $< \gamma, H >$. Also, it is clear that $< \gamma, H >$ is contained in $< \tau, \gamma, H >$. Thus, $< \tau, \gamma, H > = < \gamma, H >$.

6.2.3 Let $r_u$ and $r_{u'}$ be two reflections across lines perpendicular to $m$. Then, the composition $r_u \circ r_{u'}$ must be a translation, as these lines will be parallel. Thus, $r_u \circ r_{u'} = T^k$ for some $k$, and $r_{u'} = r_u \circ T^k$.

6.2.5 Consider $g^2$. This must be a translation, so $g^2 = T_{kv}$ for some $k$ where $T_{kv}$ is the fundamental translation. Then, $g = T_{kv} \circ r_m$ where $m$ is the midline. Suppose $k$ is an integer, say $k = j$. Then, since $T_{(v-j)v}$ is in the group, we have $T_{(v-j)v} \circ g = T_{(v-j)v} \circ T_{kv} \circ r_m = T_{v} \circ r_m$ is in the group.

Otherwise, $k = j + \frac{1}{2}$ for some integer $j$. We can find $T_{-jv}$ in the group such that $T_{-jv} \circ g = T_{v} \circ r_m$ is in the group.

6.2.7 The composition $r_v \circ r_u$ must be a translation. Also, $r_v \circ r_u (A) = r_v (A) = C$, then the translation vector must be $\vec{AC}$. But, the length of $\overline{AC}$ is twice that of $\overline{AB}$. So, we get that $2\overline{AB} = k'v$ for some $k'$. Now, either $k'$ is even or it is odd. The result
follows.

**6.2.9** From Table 4.1 we know that $\tau \circ H$ or $H \circ \tau$ is either a translation or a rotation, so it must be either $\tau^k$ for some $k$ or $H$ for $A$ on $m$. Thus, any composition of products of $\tau$ and $H$ can be reduced ultimately to a simple translation or half-turn, or to some $\tau^j \circ H_B$ or $H_B \circ \tau_j$, which are both half-turns. Thus, the subgroup generated by $\tau$ and $H$ cannot contain $r_m$ or $r_u$ or $\gamma$ and none of $<\tau, r_m>$ or $<\tau, r_u>$ or $<\tau, r_m>$ can be subgroups of $<\tau, H>$.

**6.2.11** The compositions $\tau^k \circ r_m$ or $r_m \circ \tau^k$ generate glide reflections with glide vectors $kv$. The composition of $\tau$ with such glide reflections generates other glide reflections with glide vectors $(k + j)v$. The composition of $r_m$ with a glide in the direction of $m$ will generate a translation. Thus, compositions of the three types of symmetries—glides, $r_m$, and $\tau^k$—will only generate symmetries within those types. Thus, $<\tau, \gamma>$ cannot be a subgroup of $<\tau, r_m>$, since $\gamma$ has translation vector of $\frac{v}{2}$ which cannot be generated in $<\tau, r_m>$. Also, neither $<\tau, r_u>$ nor $<\tau, H>$ can be subgroups of $<\tau, r_m>$.

**6.2.13** First Row: $<\tau>$, $<\tau, \gamma>$. Second Row: $<\tau, \gamma, H>$, $<\tau, r_m>$, $<\tau, r_u>$. Third Row: $<\tau, r_m, H>$, $<\tau, H>$. Last Row: $\tau, r_m>$.

**6.3 Wallpaper Groups**

**6.3.1** The first is rectangular, the second rhombal, and the third square.

**6.3.3** The translation determined by $f^2$ will be in the same direction as $T$, so we do not find two independent directions of translation.

**6.3.5** The lattice for $G$ will be invariant under rotations about points of the lattice by a fixed angle. By the previous problem, these rotations must be half-turns. By Theorem 6.14 the lattice must be Rectangular, Centered Rectangular, or Square.

**6.3.7** Let $C$ be the midpoint of the vector $v = \overrightarrow{AB}$, where $v$ is on
of the translation vectors for $G$. Let $m_1$ be a line perpendicular to $\overrightarrow{AB}$ at $A$. Then, $T_v = r_{m_1} \circ r_{m'_1}$ where $m'_1$ is a line perpendicular to $\overrightarrow{AB}$ at the midpoint of $\overline{AB}$. But since $r_{m_1}$ is in $G$, then $r_{m_1} \circ T_v = r_{m_1}$ is in $G$. Likewise, we could find a line $m'_2$ perpendicular to the other translation vector $w = \overrightarrow{AC}$ at its midpoint, yielding another reflection $r_{m'_2}$. The formulas for these two reflections are $r_{m'_1} = r_{m_1} \circ T_v$ and $r_{m'_2} = r_{m_2} \circ T_w$.

6.3.9 In the exercise 6.3.8 we saw that the group of symmetries can be generated from reflections half-way along the translation vectors. Thus, if we reflect the shaded region, we must get another part of the pattern. Thus, three reflections of the shaded area will fill up the rectangle determined by $v$ and $w$ and the rest of the pattern will be generated by translation.

6.3.11 If $A = lv + mw$ and $B = sv + tw$, then $0 \leq s, t \leq 1$. The length between $A$ and $B$ is the length of the vector $\overrightarrow{AB} = (s-l)v + (t-m)w$. This length squared is the dot product of $\overrightarrow{AB}$ with itself, i.e., $(s-l)^2(v \cdot v) + 2(s-l)(t-m)(v \cdot w) + (t-m)^2(w \cdot w)$. If $v \cdot w > 0$, then this will be maximal when both $(s-l)$ and $(t-m)$ are maximal. This occurs when $(s-l) = 1$ and $(t-m) = 1$, which holds only if $s = 1 = t$ and $l = m = 0$. If $v \cdot w < 0$, we need $(s-l)$ to be as negative as possible, and $(t-m)$ to be as positive as possible (or vice-versa). In either case, we get values of 0 or 1 for $s, t, l, m$.

6.3.13 A single straight line would have translational symmetries of arbitrarily small size.

6.5 Project 9 - Constructing Tessellations

Tiling is a fascinating subject. If you would like to know more about the mathematics of tiling, a good supplementary source is *Tilings and Patterns*, by Grunbaum and Shephard.

A modern master of the art of tiling is M.C. Escher. A good resource for his work is Doris Schattschneider’s book *M. C. Escher, Visions of Symmetry*. 
SOLUTIONS TO EXERCISES IN CHAPTER 6

6.5.1 The symmetry group is $p4$. 
Chapter 7

Non-Euclidean Geometry

The discovery of non-Euclidean geometry is one of the most important events in the history of mathematics. The book by Boyer and Merzbach and the University of St. Andrews web site, both listed in the bibliography of the text, are excellent references for a deeper look at this history.

Solutions to Exercises in Chapter 7

In section 7.2 we see for the first time the relevance of our earlier discussion of models in Chapter 1. The change of axioms in Chapter 7 (replacing Euclid’s fifth postulate with the hyperbolic parallel postulate) requires a change of models. As you work through this section, it is important to recall that, in an axiomatic system, it is not important what the terms actually mean; the only thing that matters is the relationships between the terms.

We introduce two different models at this point to help you recognize the abstraction that lies behind the concrete expression of points and lines in these models.
7.2.2 Mini-Project: The Klein Model

It may be helpful to do the constructions (lines, etc) of the Klein model on paper as you read through the material.

7.2.1 Use the properties of Euclidean segments.

7.2.3 The special case is where the lines intersect at a boundary point of the Klein disk. Otherwise, use the line connecting the poles of the two parallels to construct a common perpendicular.

7.3 Basic Results in Hyperbolic Geometry

In this section it is important to note the distinction between points at infinity and regular points. Omega triangles share some properties of regular triangles, like congruence theorems and Pasch-like properties, but are not regular triangles—thus necessitating the theorems found in this section.

7.3.1 Use the interpretation of limiting parallels in the Klein model.

7.3.3 First, if \( m \) is a limiting parallel to \( l \) through a point \( P \), then \( r_l(m) \) cannot intersect \( l \), as if it did, then \( r_l^2(m) = m \) would also intersect \( l \). Now, drop a perpendicular from \( r_l(P) \) to \( l \) at \( Q \), and consider the angle made by \( Q, r_l(P), \) and the omega point \( \Omega \). If there were another limiting parallel \( (n) \) to \( l \) through \( r_l(P) \) that lies within this angle, then by reflecting back by \( r_l \) we would get a limiting parallel \( r_l(n) \) that lies within the angle made by \( Q, r_l(P), \) \( P \) and the omega point of \( l \), which is impossible. Thus, \( r_l(m) \) must be limiting parallel to \( l \) and reflection maps omega points to omega points, as \( r_l \) maps limiting parallels to \( l \) to other limiting parallels. Also, it must fix the omega point, as it maps limiting parallels on one side of the perpendicular dropped to \( l \) to limiting parallels on that same side.

7.3.5 Let \( P \) be the center of rotation and let \( l \) be a line through \( P \) with the given omega point \( \Omega \). (Such a line must exist as \( P \) must correspond to a limiting parallel line \( m \), and there is always a limiting parallel to \( m \) through a given point \( P \).) Then, we can write...
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\[ R = r_n \circ r_l \] for another line \( n \) passing through \( P \). But, since \( r_l \) fixes \( \Omega \), and \( R \) does as well, then, \( r_n \) must fix \( \Omega \). But, if \( n \) and \( l \) are not coincident, then \( n \) is not limiting parallel to \( l \) and thus cannot have the same omega points as \( l \). By the previous exercise, \( r_n \) could not fix \( \Omega \). Thus, it must be the case that \( n \) and \( l \) are coincident and \( r_n = r_l \) is the identity.

7.3.7 Let \( PQ\Omega \) be an omega triangle and let \( R \) be a point interior to the triangle. Drop a perpendicular from \( Q \) to \( \overrightarrow{P\Omega} \) at \( S \). Then, either \( R \) is interior to triangle \( QPS \), or it is on \( QS \), or it is interior to \( \angle QSO \). If it is interior to \( \triangle QPS \) it intersects \( \overrightarrow{P\Omega} \) by Pasch's axiom for triangles. If it is on \( QS \) it obviously intersects \( \overrightarrow{P\Omega} \). If \( R \) is interior to \( \angle QSO \), it intersects \( \overrightarrow{P\Omega} \) by the definition of limiting parallels.

\[ \text{Figure 7.1:} \]

7.3.9 Let \( l \) be the line passing through \( R \). Then, either \( l \) passes within Omega triangle \( PR\Omega \) or it passes within \( QR\Omega \). In either case we know by Theorem 7.5 that \( l \) must intersect the opposite side, i.e. it must intersect \( \overrightarrow{P\Omega} \) or \( \overrightarrow{Q\Omega} \).

7.3.11 Suppose we had another segment \( P'Q' \) with \( \overrightarrow{PQ} \cong \overrightarrow{P'Q'} \) and let \( l' \) be a perpendicular to \( \overrightarrow{P'Q'} \) at \( Q' \). Let \( P'R' \) be a limiting parallel to \( l' \) at \( P' \). Then, by Theorem 7.8, we know that \( \angle QPR = \angle Q'R'P' \) and thus, the definition of this angle only depends on \( h \) and the length of \( \overrightarrow{PQ} \).

7.3.13 Suppose \( a(h) = a(h') \) with \( h \neq h' \). We can assume that \( h < h' \). But, then the previous exercise would imply that \( a(h) > a(h') \). Thus, if \( a(h) = a(h') \) then \( h = h' \).
7.4 Project 10 - The Saccheri Quadrilateral

As you do the computer construction of the Saccheri Quadrilateral, you may experience a flip of orientation for your construction when moving the quad about the screen. The construction depends on the orientation of the intersections of circles and these may switch as the quad is moved. A construction of the Saccheri quad that does not have this unfortunate behavior was searched for unsuccessfully by the author. A nice challenge problem would be to see if you can come up with a better construction. If you can, the author would love to hear about it!

7.4.1 Show that \( \triangle ADB \) and \( \triangle BCA \) are congruent, and then show that \( \triangle ADC \) and \( \triangle BDC \) are congruent.

![Figure 7.2: Saccheri Quadrilateral](image)

7.5 Lambert Quadrilaterals and Triangles

7.5.1 Referring to figure 7.6, we know \( \triangle ACB \) and \( \triangle ACE \) are congruent by SAS. Thus, \( \angle ACB \cong \angle ECA \). Since \( \angle ACD \cong \angle FCA \) and both are right angles, then \( \angle BCD \cong \angle FCE \). Then, \( \triangle BCD \) and \( \triangle FCE \) are congruent by SAS. We conclude that \( BD \cong FE \) and the angle at \( E \) is a right angle.

7.5.3 Create two Lambert quadrilaterals from the Saccheri quadrilateral, and then use Theorem 7.13.

7.5.5 Since the angle at \( O \) is acute, then \( OAA' \) and \( OBB' \) are triangles. Also, since \( OA < OB \), then \( A \) is between \( O \) and \( B \), and likewise \( A' \) is between \( O \) and \( B' \). Thus, the perpendicular \( n \) at \( A \) to \( AA' \) will enter \( \triangle OBB' \). By Pasch’s axiom it must intersect \( \overline{OB'} \) and...
SOLUTIONS TO EXERCISES IN CHAPTER 7

$BB'$. It cannot intersect $OB'$ as $n$ and $OB'$ must be parallel. Thus, $n$ intersects $OB'$ at $C$. Then, $A'ACB'$ is a Lambert Quadrilateral and $B'C > A'A$. Since $C$ is between $B$ and $B'$ we have $B'B > A'A$.

7.5.7 Let $m$ be right limiting parallel to $l$ at $P$ and let $P'$ be a point on $m$ to the right of $P$ (i.e. in the direction of the omega point). Let $Q$ and $Q'$ be the points on $l$ where the perpendiculars from $P$ and $P'$ to $l$ intersect $l$.

We claim that $m\angle QPP' < m\angle Q'P'R$, where $R$ is a point on $m$ to the right of $P'$. If these angles were equal we would have $PQ \cong P'Q'$ by Exercise 7.3.11, and thus $QPP'Q'$ would be a Saccheri quadrilateral, which would imply that $\angle Q'P'R$ is a right angle, which is impossible. If $m\angle QPP' > m\angle Q'P'R$, then $PQ < P'Q'$ by exercise 7.3.12, which would imply that we could find a point $S$ on $P'Q'$ with $PQ = Q'S$, yielding Saccheri quadrilateral $PQQ'S$. Then, $\angle PSC$ must be acute, which contradicts the Exterior angle theorem for $\triangle PSC$.

Thus, $m\angle QPP' < m\angle Q'P'R$, and the result follows from exercise 7.3.12.

7.5.9 If they had more than one common perpendicular, then we would have a rectangle.

7.5.11 Suppose Saccheri Quadrilaterals $ABCD$ and $EFGH$ have $AB \cong EF$ and $\angle ADC \cong \angle EHG$. If $EH > AD$ then we can find $I$ on $EH$ and $J$ on $FG$ such that $EI \cong FJ \cong AD$. Then, by repeating
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application of SAS on sub-triangles of $ABCD$ and $EIJF$ we can show that these two Saccheri Quadrilaterals are congruent. But this implies that the angles at $H$ and $I$ in quadrilateral $IHGJ$ are supplementary, as are the angles at $G$ and $J$, which means that we can construct a quadrilateral with angles sum of 360. This contradicts Theorem 7.15, by considering triangles created by a diagonal of $IHGJ$.

![Figure 7.4](image)

7.5.13 No. To construct a scale model, we are really constructing a figure similar to the original. That is, a figure with corresponding angles congruent, and length measurements proportional by a non-unit scale factor. But, Theorem 7.18 implies that any such scale model must have lengths preserved.

7.6 Area in Hyperbolic Geometry

In this section we can refer back to the mini-project we did on area in Chapter 2. That discussion depended on rectangles as the basis for a definition of area. In hyperbolic geometry, no rectangles exist so the next best shape to base area on is the triangle. This explains the nature of the theorems in this section.

7.6.1 Let $J$ be the midpoint of $AB$ and suppose that $EF$ cut $AB$ at some point $K \neq J$. Then, on $EJ$ we can construct a second Saccheri Quadrilateral by the method of dropping perpendiculars from $B$ and $C$ to $EJ$. Now, $BC$ is the base of the original Saccheri Quadrilateral $BCIH$ and the new Saccheri Quadrilateral. Thus, if
is the perpendicular bisector of $BC$, then $n$ meets $\overrightarrow{E''F}$ and $\overrightarrow{E''J}$ at right angles. Since $E''$ is common to both curves, we get a triangle having two right angles, which is impossible.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.5.png}
\caption{Figure 7.5:}
\end{figure}

7.6.3 This question can be argued both ways. If we could make incredibly precise measurements of a triangle, then we could possibly measure the angle sum to be less than 180. However, since the universe is so vast, we would have to have an incredibly large triangle to measure, or incredibly good instruments. Also, we could never be sure of errors in the measurement overwhelming the actual differential between the angle sum and 180.

7.7 Project 11 - Tiling the Hyperbolic Plane

A nice artistic example of hyperbolic tilings can be found in M. C. Escher’s Circle Limit figures. Consult Doris Schattschneider’s book *M. C. Escher, Visions of Symmetry* for more information about these tilings.

7.7.1 Reasoning as we did on Page 261 of the text, we see that if we have $k$ regular $n$-gons meeting at a common vertex, then

\[180n < 360 + 2n\alpha\]
CHAPTER 7. NON-EUCLIDEAN GEOMETRY

where $\alpha = \frac{360}{2k}$. Then,

$$\frac{360}{k} > 180 = \frac{360}{n}$$

and dividing by 360 and re-arranging gives

$$\frac{1}{n} + \frac{1}{k} > \frac{1}{2}$$

Thus, since $\frac{1}{3} + \frac{1}{3} > \frac{1}{2}$ we have that a (3, 3) tiling is possible.

7.7.3 In a (6, 5) tiling we have regular hexagons meeting 5 at a vertex. The interior angles of the hexagons must be $\frac{360}{5} = 72$. Triangulating such a hexagon by triangles to the center, we see that the central angle must be 60 degrees and the base angles of the isosceles triangles must be 36 degrees (half the interior angle).

Thus, to build the tiling we start with a triangle of angles 60, 36, and 36 and continue the construction just as we did in the lab.
Chapter 8

Non-Euclidean Transformations

In this chapter we extend our notion of isometry from Euclidean geometry to hyperbolic geometry. The discussion on pages 316-318 is intended to make the subsequent focus on Möbius transformations a natural condition for carrying out this extension.

Section 8.1 might seem to be a side-track, but it is necessary groundwork material needed to put the subsequent development of isometries on a firm footing.

Solutions to Exercises in Chapter 8

8.2 Isometries in the Poincaré Model

In this section we see what isometries look like in the Poincaré Model. We use the principles of Klein’s Erlanger Programm here. That is, we are able to prove general results about figures by transforming the figures to “nice” locations and proving the result there.

8.2.1 Let $G$ be the set of rigid motions. Let $f(z) = e^{i\phi_1}z + b_1$ and $g(z) = e^{i\phi_2}z + b_2$. Then, $g \circ f(z) = e^{i\phi_2}(e^{i\phi_1}z + b_1) + b_2 = e^{i\phi_1 + \phi_2}z + (e^{i\phi_2}b_1 + b_2)$ and thus $g \circ f(z)$ is in $G$. Since $f^{-1}(z) = e^{-i\phi_1}z - e^{-i\phi_1}b_1$
then \( f^{-1} \) is in \( G \). If \( \phi = 0 \) and \( b = 0 \) we get the identity in \( G \). Last, associativity is automatic, as function composition is associative.

**8.2.3** By Theorem 8.3 we can find a transformation such that the points 1, -1, and \( i \) go to 1, \( \infty \), and 0. This transformation is
\[
f(z) = \frac{z - i}{z + 1} \cdot \frac{2}{1-i}.
\]
Also, \( g(z) = \frac{z - 2}{z + 1} \) takes 1, -1, and 0 to 1, \( \infty \), and 0. Then, \( g^{-1} \circ f \) is the desired transformation by Theorem 8.7.

**8.2.5** Choose \( a = 1 = d \) and \( b = c = 0 \) in \( f(z) = \frac{az + b}{cz + d} \).

**8.2.7** Because function composition is associative.

**8.2.9** Let \( P = z_0 \) and \( Q = z_1 \). Let \( f(z) = \frac{z - z_0}{1-z_0} \). Then, \( f(z_0) = 0 \) and \( f \) is a hyperbolic isometry. Next, let \( g(z) = \frac{z - z_1}{z_1} \). Then, \( g(z_1) = 0 \) and \( g \) is a hyperbolic isometry. The composition \( g^{-1} \circ f \) maps \( P \) to \( Q \).

**8.2.11** Let \( l \) be a hyperbolic line from \( P \) to \( Q \). We can find a hyperbolic transformation that maps \( P \) to the origin (see the explanation for exercise 8.2.9). If the transformed line does not lie along the axis, we can transform it to the axis by a rotation. The cross-ratio is invariant under both of these transformations. Clearly, the cross-ratio defined for points on the \( x \)-axis is real. Also, the cross-ratio will look like \( \frac{1}{1-b+1} \) which is always positive.

**8.2.13** Since \( d_H \) is invariant under hyperbolic isometries we have \( d_H(z_0, z_1) = d_H(g(z_0), g(z_1)) \). Since, \( g(z_1) = 0 \) we have by Theorem 6.12
\[
d_H(z_0, z_1) = \ln\left(\frac{1 + |g(z_0)|}{1 - |g(z_0)|}\right)
= \ln\left(\frac{1 + \frac{|z_0 - z_1|}{1 - z_0 z_1}}{1 - \frac{|z_0 - z_1|}{1 - z_0 z_1}}\right)
\]
Finding a common denominator in the last equation yields the result.

**8.2.15** As in the proof of Theorem 8.11, we know that
\[
d_P(z_0, z_1) = |\ln(\frac{z_0 - w_1 z_1 - w_0}{z_0 - w_0 z_1 - w_1})|
= |\ln((z_0, z_1, w_1, w_0))|
\]
for $z_0$ and $z_1$ in the disk, where $w_0$ and $w_1$ are the points of intersection of the circle through $z_0$ and $z_1$ (call this circle $c$) with the unit circle. Let $z_0^*$ and $z_1^*$ be the inverse points of $z_0$ and $z_1$ with respect to $c$. Then, by the proof of Lemma 8.8 we have

\[
d P(z_0^*, z_1^*) = |\ln((z_0^*, z_1^*, w_1, w_0))| = |\ln((z_0, z_1^*, w_1, w_0))|
\]

But, $(z_0, z_1^*, w_1, w_0) = (z_0, z_1^*, w_1, w_0)$, as $z_0$, $z_1$, $w_0$, and $w_1$ lie on the same circle. Thus,

\[
d P(z_0^*, z_1^*) = |\ln((z_0, z_1^*, w_1, w_0))| = |\ln((z_1^*, z_0, w_1, w_0))| = |\ln((z_1^*, z_0, w_1, w_0))| = |\ln((z_1, z_0, w_1, w_0))|
\]

Again, $(z_1, z_0, w_1, w_0) = (z_1, z_0, w_1, w_0)$ and so,

\[
d P(z_0^*, z_1^*) = |\ln((z_1, z_0, w_1, w_0))| = |\ln((z_0, z_1, w_1, w_0))| = d P(z_0, z_1)
\]

To show that inversion is a reflection across $c$, we just note that inversion preserves the circle of inversion, and thus fixes the Poincaré line defined by $c$.

### 8.3 Isometries in the Klein Model

In section 8.2 we see isometries treated in a very functional way—we have formulas for isometries in the Poincaré disk defined by complex rational functions. This section serves as a nice contrast in that isometries will be defined in a very geometric way through the use of poles. Also, isometries are defined by starting with reflections, in the same way isometries were developed in Chapter 5.
8.3.1 Let \( t \) be the Klein line through \( P \) and \( P' \), and construct the pole of \( t \). Let \( \Omega \) be the omega point where the Euclidean line through the pole of \( t \) and \( P \) meets the boundary circle (on the other side of \( t \) from the pole of \( t \)). Let \( \Omega' \) be the omega point where the Euclidean line through the pole of \( t \) and \( P' \) meets the boundary circle (on the same side of \( t \) from the pole of \( t \)). (Refer to Fig. 8.1.) Then, the point \( Q \) where \( \Omega \Omega' \) intersects \( t \) is the midpoint of \( PP' \). This can be seen by using the angle-angle congruence theorem for Omega triangles. The line \( l \) through \( Q \) and the pole of \( t \) will be the perpendicular bisector of \( PP' \).

\[
\text{Figure 8.1:}
\]

8.3.3 One possible construction is illustrated in Fig. 8.2. Let \( \overline{AB} \) be a diameter of the Klein disk and let \( A \) be a point not at the center. Let \( B \) be the reflection of \( A \) across a diameter perpendicular to \( \overline{AB} \) and construct two Klein lines (\( l \) and \( m \)) at \( A \) and \( B \) that are perpendicular to \( \overline{AB} \). Construct the poles \( C \) and \( D \) to these lines and let \( \overrightarrow{CE} \) be a ray from \( C \) intersecting \( l \) at \( E \). This ray will create
SOLUTIONS TO EXERCISES IN CHAPTER 8

Klein line \( n \). Then, there will be a common perpendicular (\( \overrightarrow{GH} \)) to \( m \) and \( n \), using the result from Ex 8.3.2. Also, this line must be on the same side of \( \overline{AB} \) as \( E \) is. Then, \( AEGHB \) is a pentagon with five right angles.

8.3.5 We know from the construction of a Klein reflection that \( r \) will map a point \( P \) to a point \( P' \) that lies on a line perpendicular to \( l \). Thus, if \( P \) is already on a perpendicular \( t \) to \( l \), then its reflection is again on \( t \). Likewise, \( r_m(r_l(P)) \) is again on \( t \).

8.4 Mini-Project: The Upper Half-Plane Model

In this project we see yet a third model for hyperbolic geometry. A significant new development in this section is the idea of model
isomorphism. It would be a good idea to review this idea before you start this project.

**8.4.1** There are two cases. If $c = 0$, then $f(z) = a'z + b'$ where $a' = \frac{a}{d}$ and $b' = \frac{b}{d}$. Since $f(0) = b'$, then $b'$ is real. Since $f(1) = a' + b''$ is real then $a'$ is real. Clearly, we can assume $d = 1$ in the fraction defining $f$. Thus, $a, b, c,$ and $d$ are real.

If $c \neq 0$, then we can again assume $c = 1$ by dividing top and bottom of the fraction by $c$. Since $f(0) = \frac{a}{d} = r_1$ is real, then $b = r_1d$. Also, since $f(\infty) = \frac{a}{c} = r_2$ is real, then $a = r_2c$. Thus, $f(z) = \frac{r_2cz + r_1d}{cz + d}$. Now, for some real $r_3$ we have $f(r_3) = \frac{r_2cr_3 + r_1d}{cr_3 + d} = \infty$. Thus, $cr_3 + d = 0$ or $d = -r_3c$. Then,

$$f(z) = \frac{r_2cz + r_1(-r_3c)}{cz - r_3c} = \frac{r_2cz - r_1r_3c}{cz - r_3c} = \frac{r_2z - r_1r_3}{z - r_3}$$

Comparing this fraction with the original we see that $a, b, c,$ and $d$ are real.

**8.4.3** If we consider the $x$-axis as the equivalent of the Poincare circle, then “lines” should be clines that meet this boundary at right angles. That is, lines should be either Euclidean lines that are perpendicular to the $x$-axis, or arcs of circles perpendicular to the axis. That is, semi-circles with centers along the axis.

**8.4.5** You can argue that any configuration of a “line” and point off the line can be transformed by a suitable upper half-plane transformation to the scene illustrated in Figure 8.4. Clearly, there are an infinite number of semi-circles through $z_0$ that do not intersect the $y$-axis.

### 8.6 Hyperbolic Calculation

In this section we do some basic calculus of hyperbolic geometry. Klein’s transformational view really shines here. We see how to do...
develop some exceptionally nice formulas for arclength, the angle of parallelism, and area using proofs based on simple configurations. Also, the hyperbolic Pythagorean Theorem is a nice result in this section. The fact that hyperbolic geometry is "locally Euclidean" can be demonstrated nicely with the hyperbolic Pythagorean Theorem. If we compute the Taylor expansion for cosh we see that \( \cosh(c) = \cosh(a) \cosh(b) \) has as its second-order approximation the Euclidean Pythagorean Theorem.

8.6.1 Use the definition of cosh and sinh.
8.6.3 This is a simple matter of checking the algebra.
8.6.5 Since the map \( S \) preserves the two distance functions in the models, then the lengths of the curves must be the same.

Next,

\[
|z'| = \left| \frac{(w+i) - (w-i)}{(w+i)^2} \right| w' = \frac{2}{|w+i|^2} |w'|
\]

Thus, using the change of variable formula for integration, we get

\[
\int_a^b \frac{2|z'(t)|}{1-|z|^2} dt = \int_a^b \frac{4|w'(t)|}{|w+i|^2} dt
\]

\[
= \int_a^b \frac{4|w'(t)|}{|w+i|^2 - |w-i|^2} dt
\]

\[
= \int_a^b \frac{4|w'(t)|}{(w+i)(w-i) - (w-i)(w+i)} dt
\]

\[
= \int_a^b \frac{4|w'(t)|}{2(-iw+iw)} dt
\]

\[
= \int_a^b \frac{|w'(t)|}{v(t)} dt
\]
8.7 Project 12 - Infinite Real Estate?

You will probably not believe the results of this project, which makes it such a great lab!

8.7.1 We note that

\[
S(w) = i \frac{w - i}{w + i} \\
= i \frac{(u + i(v - 1))}{(u + i(v + 1))} \\
= i \frac{(u + i(v - 1))(u - i(v + 1))}{u^2 + (v + 1)^2} \\
= i \frac{(u^2 + (v^2 - 1)) + i(-2u)}{u^2 + (v + 1)^2}
\]

Thus,

\[
x = \frac{2u}{u^2 + (v + 1)^2} \\
y = \frac{u^2 + (v^2 - 1)}{u^2 + (v + 1)^2}
\]

8.7.3 The angle \( \theta \) will be defined by the tangent \( \overrightarrow{PB} \) to the circle at \( P \). If \( \theta \) is 90 degrees, then this tangent is perpendicular to the \( y \)-axis and it is obvious that the angle in the \( \Omega \) triangle at \( P \) is a right angle.

Otherwise, we can assume the tangent intersects the \( x \)-axis at \( B \). It follows that \( \Delta OPB \) is a right triangle with right angle at \( B \). Drop a perpendicular from \( P \) to the \( x \)-axis, intersecting at \( A \). Then \( \angle APB \) has measure \( \theta \). It immediately follows that the interior angle of the doubly limiting triangle at \( P \) has measure \( \theta \).

Lab Conclusion For the conclusion of the lab, note that a triangular area in hyperbolic geometry has area bounded by \( \pi \) by Theorem 8.28. A 4-sided figure can be split into two triangular figures, and so its area must be bounded by \( 2\pi \). A five-sided figure would have area bounded by \( 3\pi \), etc.
Chapter 9
Fractal Geometry

Much of the material in this chapter is at an advanced level, especially the sections on contraction mappings and fractal dimension—Sections 9.5 and 9.6. But this abstraction can be made quite concrete by the computer explorations developed in the chapter. In fact, the computer projects are the only way to really understand these geometric objects on an intuitive level.

Solutions to Exercises in Chapter 9

9.3 Similarity Dimension

The notion of dimension of a fractal is very hard to make precise. In this section we present one simple way to define dimension, but there are also other ways to define dimension as well, each useful for a particular purpose and all agreeing with integer dimension, but not necessarily with each other.

9.3.1 Theorem 2.27 guarantees that the sides of the new triangle are parallel to the original sides. Then, we can use SAS congruence to achieve the result.

9.3.3 At each successive stage of the construction, 8 new squares are created, each of area $\frac{1}{9}$ the area of the squares at the previous.
CHAPTER 9. FRACTAL GEOMETRY

stage. Thus, the pattern for the total area of each successive stage of the construction is

\[ l = 1 - \frac{1}{9} - \frac{8}{81} - \frac{64}{9^3} - \ldots \]

\[ = 1 - \frac{1}{9} \sum_{k=0}^{\infty} \left(\frac{8}{9}\right)^k \]

\[ = 1 - \frac{1}{9} \frac{1}{1 - \frac{8}{9}} \]

\[ = 1 - 1 \]

\[ = 0 \]

Thus, the area of the final figure is 0.

**9.3.5** The similarity dimension would be \( \frac{\log(4)}{\log(3)} \).

**9.3.7** Split a cube into 27 sub-cubes, as in the Menger sponge construction, and then remove all cubes except the eight corner cubes and the central cube. Do this recursively. The resulting fractal will have similarity dimension \( \frac{\log(9)}{\log(3)} \), which is exactly 2.

### 9.4 Project 13 - An Endlessly Beautiful Snowflake

If you want a challenge, you could think of other templates based on a simple segment, generalizing the Koch template and the H template from exercise 9.4.4.

**9.4.1** At stage 0 the Koch curve has length 1. At stage 1 it has length \( \frac{4}{3} \). At stage 2 it has length \( \frac{16}{9} = \frac{4^2}{3^2} \), since each segment is replaced by the template, which is \( \frac{4}{3} \) as long as the original segment. Thus, at stage \( n \) the length will be \( \frac{4^n}{3^n} \), and so the length will go to infinity.

**9.4.3** The similarity dimension will be that of the template replacement fractal. The similarity ratio is \( \frac{1}{3} \) and it takes 4 sub-objects to create the template. Thus, the similarity dimension is \( \frac{\log(4)}{\log(3)} \).
9.6 Fractal Dimension

Sections 9.5 and 9.6 are quite "thick" mathematically. To get some sense of the Hausdorff metric, you can compute it for some simple pairs of compact sets. For example, two triangles in different positions. Ample practice with examples will help you get a feel for the mini-max approach to the metric and this will also help you be successful with the homework exercises.

9.6.1 A function \( f \) is continuous if for each \( \epsilon > 0 \) we can find \( \delta > 0 \) such that \( |f(x) - f(y)| < \epsilon \) when \( 0 < |x - y| < \delta \). Let \( S \) be a contraction mapping with contraction factor \( 0 \leq c < 1 \). Then, given \( \epsilon \), let \( \delta = \epsilon \) (if \( c = 0 \)) and \( \delta = \frac{\epsilon}{c} \) (if \( c > 0 \)).

If \( c = 0 \) we have \( 0 = |S(x) - S(y)| \leq |x - y| < \delta = \epsilon. \)

If \( c > 0 \), we have \( |S(x) - S(y)| \leq c|x - y| < c\frac{\epsilon}{c} = \epsilon. \)

9.6.3 Property (2): Since \( d_H(A, A) = d(A, A) \), and since \( d(A, A) = \max\{d(x, A) | x \in A \} \), then we need to show \( d(x, A) = 0 \). But \( d(x, A) = \min\{d(x, y) | y \in A \} \), and this minimum clearly occurs when \( x = y \); that is, when the distance is 0.

Property (3): If \( A \neq B \) then we can always find a point \( x \) in \( A \) that is not in \( B \). Then, \( d(x, B) = \min\{d(x, y) | y \in B \} \) must be greater than 0. This implies that \( d(A, B) = \max\{d(x, B) | x \in A \} \) is also greater than 0.

9.6.5 We know that

\[
\begin{align*}
    d(A, C \cup D) &= \max\{d(x, C \cup D) | x \in A\} \\
                 &= \max\{\min\{d(x, y) | x \in A \text{ and } y \in C \text{ or } D\}\} \\
                 &= \max\{\min\{\min\{d(x, y) | x \in A, y \in C\}, \min\{d(x, y) | x \in A, y \in D\}\} | x \in A\}
\end{align*}
\]

The last expression is clearly less than or equal to \( \max\{d(x, C) | x \in A\} = d(A, C) \) and also less than or equal to \( \max\{d(x, D) | x \in A\} = d(A, D) \).

9.6.7 There are three contraction mappings which are used to construct Sierpinski’s triangle. Each of them has contraction scale...
factor of $\frac{1}{2}$. Thus, we want $(\frac{1}{2})^D + (\frac{1}{2})^D + (\frac{1}{2})^D = 1$, or $3(\frac{1}{2})^D = 1$.

Solving for $D$ we get $D = \frac{\log(3)}{\log(2)}$.

9.7 Project 14 - IFS Ferns

Do not worry too about getting exactly the same numbers for the scaling factor and the rotations that define the fern. The important idea is that you get the right types of transformations (in the correct order of evaluation) needed to build the fern image. For exercise 9.7.5, it may be hopeful to copy out one piece of the image and then rotate and move it so it covers the other pieces, thus generating the transformations needed.

9.7.1 The rotation matrix $R$ is given by

$$
\begin{bmatrix}
\cos\left(\frac{5\pi}{180}\right) & \sin\left(\frac{5\pi}{180}\right) \\
-\sin\left(\frac{5\pi}{180}\right) & \cos\left(\frac{5\pi}{180}\right)
\end{bmatrix} \approx
\begin{bmatrix}
0.996 & 0.087 \\
-0.087 & 0.996
\end{bmatrix}
$$

The scaling matrix $S$ is given by

$$
\begin{bmatrix}
0.8 & 0 \\
0 & 0.8
\end{bmatrix}
$$

If we let $T$ be the translation in the vertical direction by $h$, then $T_1 = T \circ S \circ R$, which after rounding to the nearest tenth, matches the claimed affine transformation in the text.

9.7.3 The rotation matrix $R$ is given by

$$
\begin{bmatrix}
\cos\left(-\frac{60\pi}{180}\right) & \sin\left(-\frac{60\pi}{180}\right) \\
\sin\left(-\frac{60\pi}{180}\right) & \cos\left(-\frac{60\pi}{180}\right)
\end{bmatrix} \approx
\begin{bmatrix}
0.5 & 0.866 \\
-0.866 & 0.5
\end{bmatrix}
$$

The scaling matrix $S$ is given by

$$
\begin{bmatrix}
0.3 & 0 \\
0 & 0.3
\end{bmatrix}
$$

The reflection matrix $r$ is given by

$$
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
$$
If we let $T$ be the translation in the vertical direction by $\frac{h}{2}$, then $T_3 = T \circ S \circ R \circ r$, which after rounding to the nearest hundredth, matches the claimed affine transformation in the text.

**9.7.5** For the lower left portion of the shape, we need to scale the whole figure down by a little less than 0.5, say by 0.48. Also, we need to rotate the figure by 90 degrees and then translate back by 0.5 in the $x$-direction to put it in place. Let $T_1$ be the net transformation accomplishing this. Then $T_1(x, y) = (-0.48y + 0.5, 0.48x)$. Let $T_2$ be the transformation for the upper left portion. Then $T_2(x, y) = (0.5x, 0.5y + 0.5)$. Let $T_3$ be the transformation for the upper right portion. Then $T_3(x, y) = (0.48y + 0.5, -0.48x + 1.0)$. Finally, let $T_4$ be the transformation for the small inner part. Then $T_4(x, y) = (0.3x + 0.3, 0.3y + 0.3)$ would work.

### 9.9 Grammars and Productions

This section will be very different from anything you have done before, except for those who have had some computer science courses. The connection between re-writing and axiomatic systems is a deep one. One could view a theorem as essentially a re-writing of various symbols and terms used to initialize a set of axioms. Also, turtle geometry is a very concrete way to view re-writing and so we have a nice concrete realization of an abstract idea.

**9.9.1** Repeated use of production rule 1 will result in an expression of the form $a^nSb^n$. Then, using production rule 2, we get $a^n b^n$.

**9.9.3** The level 1 rewrite is $+RF - LFL - FR +$. This is shown in Fig. 9.1. The level 2 rewrite is $+ - LF + RFR + FL - F + RF + LFL - FR + F + RF - LFL - FR + - F - LF + RFR + FL -$. This is shown in Fig. 9.2. For the last part of the exercise, note that all interior “lattice” points (defined by the length of one segment) are actually visited by the curve. Thus, as the level increases (and we scale the curve back to some standard size) the interior points will cover space, just as the example in section 9.9 did.
9.10 Project 15 - Words Into Plants

Grammars as representations of growth is an idea that can be tied nicely with the notion of genetics from biology. A grammar is like a blueprint governing the evolution of the form of an object such as a bush, in much the same way that DNA in its expression as proteins governs the biological functioning of an organism.

9.10.1 The start symbol was rewritten twice.

9.10.3 Here’s one simple example, plus the image generated from rewriting to a level of 3 (Fig. 9.3).

Productions: \( X \rightarrow F[+X][+X][-X][-X]X \) (Use a small turn angle)
Figure 9.3:
Appendix A

Sample Lab Report

Pollie Gonn
MCS 303 Project 0
September 12, 2003
The Amazing Pythagorean Theorem

Introduction
The Pythagorean Theorem is perhaps the most famous theorem in geometry, if not in all of mathematics. In this lab, we look at one method of proving the Pythagorean Theorem by constructing a special square. Part I of this report describes the construction used in the proof and Part II gives a detailed explanation of why this construction works, that is why the construction generates a proof of the Pythagorean theorem. Finally, we conclude with some comments on the many proofs of the Pythagorean Theorem.

Part I:
To start out our investigation of the Pythagorean Theorem, we assume that we have a right triangle with legs b and a and hypotenuce c. Our first task construction is that of a segment subdivided into two parts of lengths a and b. Since a and b are arbitrary, we just create a segment, attach a point, hide the original segment, and draw two new segments as shown.
Then, we construct a square on side $a$ and a square on side $b$. The purpose of doing this is to create two regions whose total area is $a^2 + b^2$. Clever huh? Constructing the squares involved several rotations, but was otherwise straightforward.

![Figure A.1](image1)

The next construction was a bit tricky. We define a translation from B to A and translate point C to get point H. Then, we connect H to D and H to G, resulting in two right triangles. In part II, we will prove that both of these right triangles are congruent to the original right triangle.

![Figure A.2](image2)
Next, we hide segment BC and create segments BH and HC. This is so that we have well-defined triangle sides for the next step - rotating right triangle ADH 90 degrees about its top vertex, and right triangle HGC -90 degrees about its top vertex.

Part II:
We will now prove that this construction yields a square (on DH) of side length $c$, and thus, since the area of this square is clearly equal to the sum of the areas of the original two squares, we have
$a^2 + b^2 = c^2$, and our proof would be complete. By SAS, triangle HCB must be congruent to the original right triangle, and thus its hypotenuse must be $c$. Also, by SAS, triangle DAH is also congruent to the original triangle, and so its hypotenuse is also $c$. Then, angles AHD and CHG(= ADH) must sum to 90 degrees, and the angle DHG is a right angle. Thus, we have shown that the construction yields a square on DH of side length $c$, and our proof is complete.

**Conclusion:**

This was a very elegant proof of the Pythagorean Theorem. In researching the topic of proofs of the Pythagorean Theorem, we discovered that over 300 proofs of this theorem have been discovered. Elisha Scott Loomis, a mathematics teacher from Ohio, compiled many of these proofs into a book titled *The Pythagorean Proposition*, published in 1928. This tidbit of historical lore was gleaned from the Ask Dr. Math website (http://mathforum.org/library/drmath/view/62539.html). It seems that people cannot get enough of proofs of the Pythagorean Theorem.
Appendix B

Sample Lab Grading Sheets

Sample Grade Sheet for Project 1 - The Ratio Made of Gold

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar

- 50 points - Discussion of Project Work and Solutions to Exercises
  - 5 Discussion of the Construction of the Golden Ratio
  - 5 Discussion of the Construction of the Golden Rectangle
  - 10 Solution to Exercise 1.3.1
  - 10 Solution to Exercise 1.3.2
  - 10 Solution to Exercise 1.3.3
– 10 Solution to Exercise 1.3.4

• Total Points for Project (out of 60 possible)
Sample Grade Sheet for Project 2 - A Concrete Axiomatic System

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar

- 50 points - Discussion of Project Work and Solutions to Exercises
  - 10 Discussion of Euclid’s Five Postulates
  - 10 Construction of Rectangles
  - 10 Sum of Angles in a Triangle
  - 10 Euclid’s Equilateral Triangle Construction
  - 10 Perpendicular to a Line through a Point Not on the Line

- Total Points for Project (out of 60 possible)
Sample Grade Sheet for Project 3 - Special Points of Triangle

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar

- 60 points - Discussion of Project Work and Solutions to Exercises
  - 10 Discussion of Work Done in Lab
  - 10 Exercise 2.3.1
  - 10 Exercise 2.3.2
  - 10 Exercise 2.3.3
  - 10 Exercise 2.3.4
  - 10 Exercise 2.3.5

- Total Points for Project (out of 70 possible)
Sample Grade Sheet for Project 4 - Circle Inversion and Orthogonality

• 10 points - Organization And Writing Mechanics
  – 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and conclusion.
  – 5 Spelling and Grammar

• 50 points - Discussion of Project Work and Solutions to Exercises
  – 10 Discussion of Work Done in Lab
  – 10 Exercise 2.7.1
  – 10 Exercise 2.7.2
  – 10 Exercise 2.7.3
  – 10 Exercise 2.7.4

• Total Points for Project (out of 60 possible)
APPENDIX B. SAMPLE LAB GRADING SHEET

Sample Grade Sheet for Project 7 - Quilts and Transformations

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar

- 50 points - Discussion of Project Work and Solutions to Exercises
  - 10 Discussion of Initial Work Done on Quilt 1
  - 10 Exercise 4.5.1
  - 10 Exercise 4.5.2
  - 10 Exercise 4.5.3
  - 10 Exercise 4.5.4

- Total Points for Project (out of 60 possible)
Sample Grade Sheet for Project 8 - Constructing Compositions

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar

- 50 points - Discussion of Project Work and Solutions to Exercises
  - 10 Discussion of Initial Work Done
  - 10 Exercise 4.8.1
  - 10 Exercise 4.8.2
  - 10 Exercise 4.8.3
  - 10 Exercise 4.8.4

- Total Points for Project (out of 60 possible)
Sample Grade Sheet for Project 9 - Constructing Tessellations

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate
    headings and captions, including an introduction and
    conclusion.
  - 5 Spelling and Grammar

- 30 points - Discussion of Project Work and Solutions to Exercises
  - 10 Discussion of Initial Work Done
  - 10 Exercise 5.5.1
  - 10 Exercise 5.5.2

- Total Points for Project (out of 40 possible)
Sample Grade Sheet for Project 10 - The Saccheri Quadrilateral

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar

- 60 points - Discussion of Project Work and Solutions to Exercises
  - 10 Discussion of Initial Work Done
  - 10 Exercise 6.5.1
  - 8 Exercise 6.5.2 part i
  - 8 Exercise 6.5.2 part ii
  - 8 Exercise 6.5.2 part iii
  - 8 Exercise 6.5.2 part iv
  - 8 Exercise 6.5.2 part v

- Total Points for Project (out of 70 possible)