CONVEX *tangrams*

PAUL
SCOTT

Tangrams

The Chinese tangram puzzle was known as far back as 1813. It has remained popular ever since. It consists of seven simple polygonal pi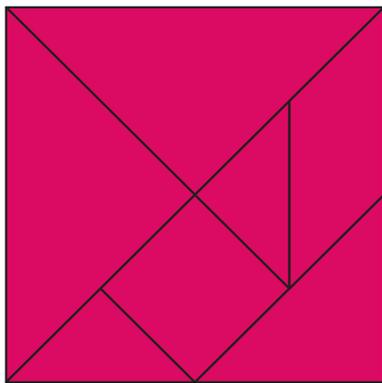eces of card which can be assembled in the form of a square as shown in Figure 1. The reader is presented with some popular shape such as the man or cat above, and then asked to construct this using the tangram pieces. There are whole books of such shapes, and one can spend hours finding the various solutions.

Figure 1

In the past I have only had a cursory interest in tangrams. Their geometrical nature attracts me, but not the prospect of constructing endless arbitrary shapes. However, my interest was sparked by the following question: How many *convex* tangrams can be formed?

Such shapes will be convex polygons, which can be thought of as polygons with no re-entrant angles. Alternatively, we can use the usual convexity definition: each two points of the figure can be joined by a line segment which lies completely within the figure.

I find this type of problem really appealing. It is elementary in the sense that anyone can understand it. It appears to be a mathematical problem of some worth. The solution is by no means obvious or easy to find. On the other hand the problem lends itself to a lot of exper-imentation and "playing around". You might like to set it as a class project!

- How many convex tangrams can you find?
- Are there convex tangrams other than the square and rectangle shown above?
- What constraints can you find on the size of the final figures?
- What constraints can you find on the configuration of the pieces?

You might like to stop reading here temporarily so that you can work alongside your class on an equal footing.

Rational and irrational

Let us look at the relative side-lengths of the tangram pieces. We take the edge-length of the square piece to be 1 (Figure 2).
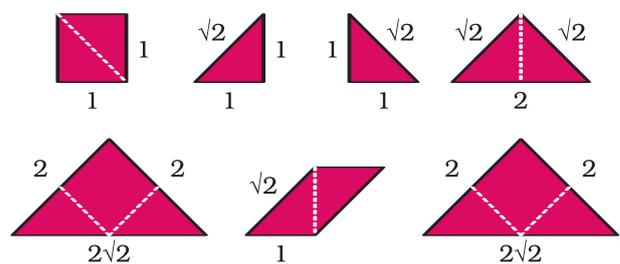

Figure 2

We notice that the edges are of two types: rational (1, 2) and irrational ($\sqrt{2}$, $2\sqrt{2}$). In fact each tangram piece is the union of a number of 1–1–$\sqrt{2}$ basic triangles which have edges of the same two types. Notice that for basic triangle or tangram piece, edges adjacent to a right angle are of the same type; edges adja-

cent to a 45° or 135° angle are of opposite types.

Now this raises an interesting question. If we look at the tangram square and rectangle, we see that all the rational sides lie in two perpendicular directions, as do all the irrational sides, and the two sets are inclined at an angle of 45°. Does this hold true for all completed tangrams? A quick look at the cat and man tangrams gives the answer in the negative; but does it hold true for all *convex* tangrams?

It is easy to convince ourselves that this is indeed the case. Suppose we have two basic triangles that have a rational and an irrational edge placed together on the common line *L*. Building up the completed triangulated convex figure from the left, being sure to fill all spaces, we note that

- no pieces cross the line *L*;
- it is impossible to complete satisfactorily the figure with a rational contribution along *L* on one side, and an irrational contribution along the other.
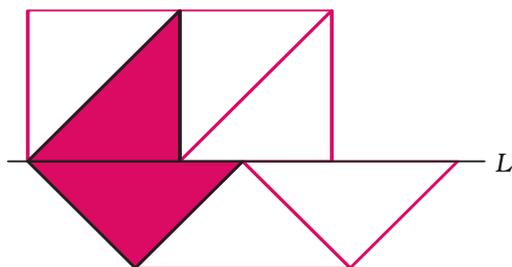

Figure 3

We deduce that it is impossible to construct a convex polygon in this way using tangram pieces, for if it were, we would have immediately a corresponding array of triangles, which we have seen is impossible.

Hence for convex tangram polygons the rational and irrational edges lie in two distinct perpendicular sets, angled at 45°.

Possible polygons

Suppose our constructed *n*-gon has *s* angles of 45°, *r* right angles, and *l* angles of 135°. Then

$$s + r + l = n,$$

and

$$45s + 90r + 135l = (n - 2)180.$$

Eliminating *l* from these two equations gives

$$2s + r = 8 - n.$$

Since $s \geq 0$, $r \geq 0$, we immediately determine that $n \leq 8$. We can now list out the possibilities:

n	8	7	6	6	5	5	4	4	4	3
s	0	0	0	1	0	1	0	1	2	2
r	0	1	2	0	3	1	4	2	0	1
l	8	6	4	5	2	3	0	1	2	0

Notice that these ten possibilities do not determine the number of possible convex tangram polygons: we already have two candidates for $(n, s, r, l) = (4, 0, 4, 0)$.

Suppose now that our convex tangram polygons are placed with their rational edges (component rational segments) horizontal and vertical. Then our list above shows that all such polygons occur in the form of a rectangle with corners possibly truncated at angles of 45°. In the figure below, the rectangle edge lengths *x*, *y*, and the edge lengths *a*, *b*, *c*, and *d* of the truncated triangular corners are all integers.
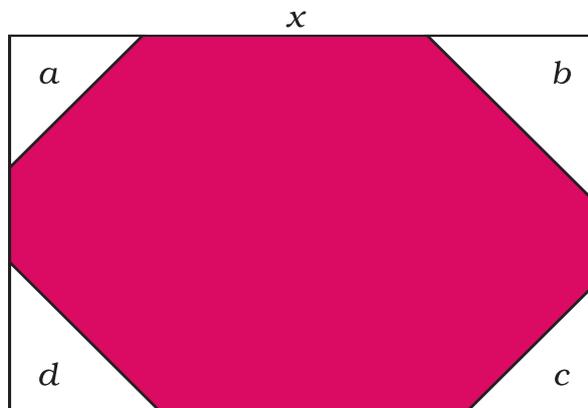

Figure 4

We may take $x \geq y$, and obviously have $a + b \leq x$, $c + d \leq x$, $a + d \leq y$, and $b + c \leq y$. We may assume, by rotating or reflecting the rectangle, that a = min(a, b, c, d). Adding the maximum lengths of the tangram pieces shows that x is well short of 20. We easily calculate (by summing the areas of the tangram pieces) that the area of the convex tangram polygon is 8. Hence

$$2xy - (a^2 + b^2 + c^2 + d^2) = 16.$$

We can now run the rather rough and ready Pascal program below to find the possible solutions x, y, a, b, c and d.

```
program tangram; {to find all possible convex
tangrams}

    var
        a, b, c, d, x, y, M: Integer;

    begin {program}

        for x:= 2 to 20 do
        for y:= 2 to x do
        for a:= 0 to y do
        for b:= a to y do
        for c:= a to y do
        for d:= a to y do

        begin{for}
            M:= 2*x*y - (a*a + b*b + c*c + d*d);
            if (a + b <= x) and (c + d <= x) and
            (a + d <= y) and (b + c <= y) and
            (M = 16) then
            writeln(x, y, a, b, c, d);
        end{for};
    end. {program}
```

This program gives the (annotated) printout shown in Figure 5. The 13 rows with "Tangram" alongside turn out to be the realis-able sets. The tangrams are pictured in Figure 6.

There are now ten cases of which to dispose. We note that case 3 is equivalent to case 1, 8 and 9 are equivalent to 7, 12 and 13 are equivalent to 11, and 18 is equivalent to 17. These cases could have been removed by more careful (and complicated) programming. Case 23 is a long thin parallelogram, too narrow to accommodate the tangram pieces, so can be discarded. Case 19 requires a $10\sqrt{2}$ length of irrational edges, which is the total length of irrational edges of the tangram

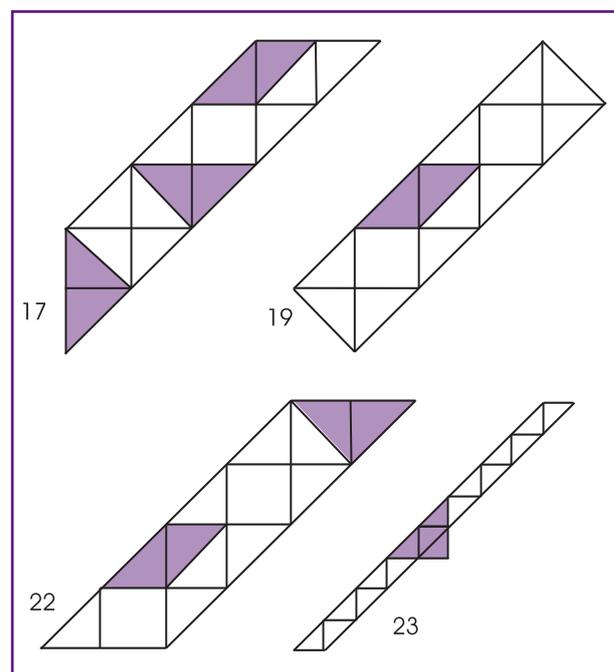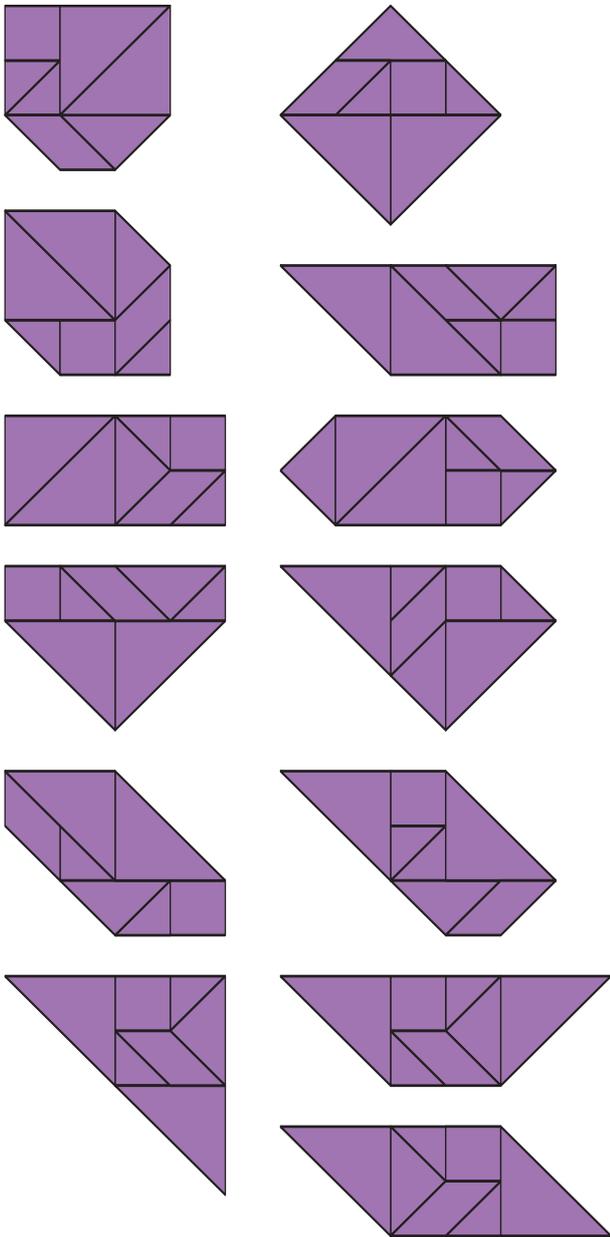|	x	y	a	b	c	d	
1.	3	3	0	0	1	1	Tangram
2.	3	3	0	1	0	1	Tangram
3.	3	3	0	1	1	0	
4.	4	2	0	0	0	0	Tangram
5.	4	3	0	0	2	2	Tangram
6.	4	3	0	2	0	2	Tangram
7.	4	4	0	0	0	4	Tangram
8.	4	4	0	0	4	0	
9.	4	4	0	4	0	0	
10.	4	4	2	2	2	2	Tangram
11.	5	2	0	0	0	2	Tangram
12.	5	2	0	0	2	0	
13.	5	2	0	2	0	0	
14.	5	2	1	1	1	1	Tangram
15.	5	3	0	1	2	3	Tangram
16.	5	3	0	2	1	3	Tangram
17.	5	5	0	3	0	5	
18.	5	5	0	5	0	3	
19.	5	5	1	4	1	4	
20.	6	2	0	0	2	2	Tangram
21.	6	2	0	2	0	2	Tangram
22.	6	4	0	4	0	4	
23.	9	8	0	8	0	8	

Figure 5

Figure 6

Figure 7

pieces; however, in this shape, both irrational edges of the parallelogram cannot be used. We can therefore discard this case.

Cases 17 and 22 can be disposed of in similar fashion. Each requires eight irrational boundary edges. A further two internal irrational edges are taken up as in Case 19 with the placing of the parallelogram. This uses up all the available irrational edges. But now consider placing the middle-sized triangle. However it is placed, a further new internal irrational edge is introduced. This impossibility means we can discard these two cases. We are thus left with the result:

There are exactly 13 convex tangrams.

These are depicted in printout order in Figure 7. This is by no means an original result. It was first proved in 1942 by Fu Traing Wang and Chuan-Chih Hsiung, and the above proof is based on their method.

Bibliography

Elffers, J. (1976). *Tangrams*. Penguin .

Fu Traing Wang, Chuan-Chih Hsiung. A theorem of the tangram. *The American Mathematical Monthly, 49*, 596–599.

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Paul Scott
Adelaide, South Australia
mail@paulscott.info