A THEOREM ON THE TANGRAM

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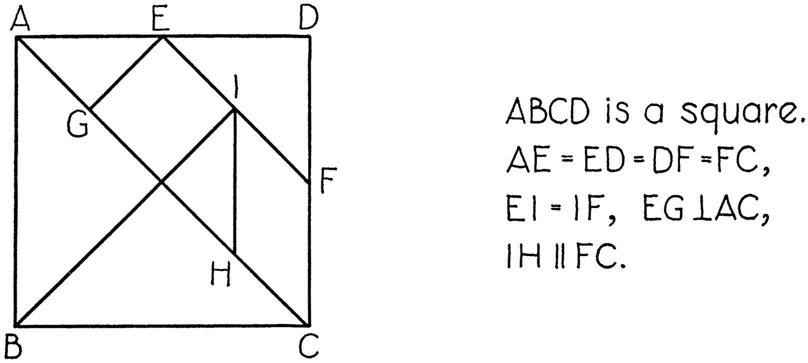
1. Introduction. The tangram is a Chinese puzzle consisting of a square card or board cut by straight incisions into different-sized pieces (five triangles, a square, and a lozenge) as shown in the following figure. These seven pieces may be combined to form many different figures. It is natural to inquire



how many convex polygons may be formed by the tangram. We propose in the present note to give a solution of this problem by proving the following theorem:

THEOREM. *By means of the tangram exactly thirteen convex polygons can be formed.*

2. Lemmas. It is easily seen that the tangram can be divided into sixteen equal isosceles right triangles. For the sake of convenience, we call the legs and the hypotenuses of these right triangles respectively *the rational* and *the irrational sides*. Then we may prove our theorem by finding out first the convex polygons formed by these sixteen triangles and then discarding those cases impossible for the tangram. For this purpose we introduce the following four lemmas.

LEMMA 1. *If sixteen equal isosceles right triangles are combined into a convex polygon, then a rational side of one triangle does not lie along an irrational side of another.*

Proof: First of all, let us suppose that of the sixteen given triangles two, denoted by ABC and $A'B'C'$, are arranged so that the irrational side $A'C'$ of the triangle $A'B'C'$ lies along the rational side AB of the triangle ABC. Since the given triangles are combined into a convex polygon, we may, without loss of generality, further suppose that the vertex A' coincides with the vertex A. In this case, at least another pair of the given triangles, denoted by DEF and $D'E'F'$, is such that one rational side $D'E'$ of the triangle $D'E'F'$ lies along the irrational side DF of the triangle DEF, and $D\equiv B$, $D'\equiv C'$, $E'\equiv F$. If we fill the angle CDE or $B'D'F'$ with one or more of the given triangles, the case where one rational side of one triangle lies along the irrational side of another triangle will occur again. Repeatedly applying the above discussion, it may be easily seen that the polygon formed by the given triangles can not be convex. This contradicts the hypothesis, and establishes the lemma.

From Lemma 1, follows immediately the following lemma:

LEMMA 2. *If sixteen equal isosceles right triangles are combined into a convex polygon, then the sides of the polygon are formed by sides of the same kind (rational or irrational) of the triangles. Moreover, if a side of the polygon which is formed by the rational or the irrational sides of the triangles is said to be a rational or an irrational side (respectively) of the polygon, then in general the rational and the irrational sides of the polygon alternate. In particular, if an angle of the polygon is a right angle, the two adjacent sides are both rational or both irrational.*

LEMMA 3. *If sixteen equal isosceles right triangles are combined into a convex polygon, then the number of the sides of the polygon does not exceed eight.*

Proof: Since the sum of all the angles of a convex polygon of n sides is equal to $(n-2)\pi$, and the maximal value of the angles formed by the given triangles is $3\pi/4$, we have $(n-2)\pi \leqq 3\pi n/4$. It follows that $n \leqq 8$.

Since the angles of the convex polygon formed by the given triangles are $3\pi/4$, $\pi/2$, or $\pi/4$, by means of Lemma 2 and Lemma 3 we easily obtain

LEMMA 4. *If sixteen equal isosceles right triangles are combined into a convex polygon, then this polygon can be inscribed in a rectangle with all the rational or the irrational sides of the polygon as the sides of the rectangle.*

3. Proof of the theorem. For the purpose of proving our theorem, we have to find out the convex polygons formed by sixteen equal isosceles right triangles. First of all, we may assume, that this convex polygon is an octagon, denoted by $ABCDEFGH$. From Lemma 2 and Lemma 4 we may, further, assume that this polygon is inscribed in a rectangle $PQRS$ and that all the rational sides BC, DE, FG, HA of the polygon lie along the sides PQ, QR, RS, SP of the rectangle respectively. If the number of irrational sides of the given triangles on AB, CD, EF, GH are respectively a, b, c, d, and PQ, QR have equal lengths, respectively, with the lines composed of x and y rational sides of the given triangles, then a, b, c, d, x, y satisfy the equation

$$a^2 + b^2 + c^2 + d^2 = 2xy - 16, \tag{1}$$

with the conditions

$$\begin{cases} a + b \leqq x, & c + d \leqq x, \\ a + d \leqq y, & b + c \leqq y, \end{cases} \tag{2}$$

Hence our problem is reduced merely to finding the integral solutions of the equation (1) and the inequalities (2). For this purpose, we denote $x=\alpha$, $y=\beta$ by (α, β), and divide our discussion into the following cases, which may be easily proved to be sufficient.

a. *The case* $y>x$, $y>5$.

(i) $x>1$. Noticing that

$$9/x + x < 2 + x \leqq y + 1, \quad \text{whenever} \quad x \geqq 5,$$

we easily obtain

$$x(y+1) > x^2 + 9, \quad \text{for} \quad x > 1 \quad \text{and} \quad y > 5. \tag{3}$$

By means of (1), (3) and the inequality $c^2+d^2 \leqq (c+d)^2 \leqq x^2$, follows immediately

$$a^2 + b^2 > (x-1)^2 + 1. \tag{4}$$

It follows that a and b are not both zero. On the other hand a and b are not both different from zero, for if $a \geqq 1$, $b \geqq 1$, then from the first inequality of (2),

$$a^2 + b^2 \leqq (a + b - 1)^2 + 1 \leqq (x-1)^2 + 1, \quad \text{for} \quad x > 1, \tag{5}$$

which contradicts (4). Whence either a or b (and not both) equals zero. Let, for instance, $b=0$; then $a \leqq x$. If, further, $a<x$, then $a \leqq x-1$, which contradicts (4). Therefore $a=x$.

Similarly, we know that $c=0$, $d=x$, or $c=x$, $d=0$. Thus we can easily show that (2, 6), (4, 6), (8, 9) are solutions.

(ii) $x=1$. In this case $a+b \leqq 1$, $c+d \leqq 1$. Therefore $a=b=c=d=0$, or $a=c=1$, $b=d=0$, or $a=c=0$, $b=d=1$, and hence we obtain the solutions (1, 8), (1, 9).

b. *The case* $x=y$. In this case we shall prove that $x \leqq 5$. First of all, it is easily seen that if $a=b=c=d=0$, there is no solution. Secondly, if $a, b, c, d<x$, then it is seen that

$$a^2 + b^2 \leqq (x-1)^2 + 1, \qquad c^2 + d^2 \leqq (x-1)^2 + 1. \tag{6}$$

From (1) and (6), we obtain

$$2x^2 - 16 \leqq 2(x-1)^2 + 2,$$

which shows $x \leqq 5$.

Thirdly, we consider the case where one of a, b, c, d is equal to x. If, for instance, $a=x$, then from (2), $b=0$ and $d=0$. Whence (1) becomes $x^2=16+c^2$, which gives $x=5$ or 4.

Finally, it is not difficult to show that when $a=b=0$ or $c=d=0$, then $x<4$; and when $a=b=c=0$, then $x=d=4$.

c. *The case* $5 \geqq y > x$. In this case we may test for every set of integral values of x, y, a, b, c, d directly from the equation (1) and the inequalities (2) and easily obtain the required solutions.

In conclusion, we summarize the complete solution of our problem related to sixteen equal isosceles right triangles as follows:

x	y	a	b	c	d
*1	8	0	0	0	0
*1	9	1	0	1	0
*1	9	1	0	0	1
*8	9	8	0	8	0
*4	6	4	0	4	0
2	6	2	0	2	0
2	6	2	0	0	2
*5	5	4	1	4	1
*5	5	5	0	3	0
3	5	3	0	1	2
3	5	3	0	2	1
2	5	1	1	1	1
2	5	2	0	0	0
4	4	2	2	2	2
4	4	4	0	0	0
3	4	2	0	2	0
3	4	2	0	0	2
2	4	0	0	0	0
3	3	1	0	1	0
3	3	1	0	0	1

It is easy to show that the solutions indicated by asterisks in the above table are useless if the given sixteen equal isosceles right triangles are supposed to form a tangram. Thus we have completely proved our theorem.

4. Remark. It should be noted that the thirteen convex polygons (four hexagons, two pentagons, six quadrangles, and a triangle) obtained in the previous section are familiar to us. We can arrange the tangram into any of them by one or more methods, but space does not permit their inclusion here.

Moreover, in these thirteen convex polygons the perimeters of the first, the second, and the eighth, according to the order in the above table, are maximum and that of the last two are minimum.