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On the Multiplicity of Polyabolos and Tangrams with Four-Fold Symmetry

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Many of us fondly remember playing Tangrams [1, 6, 10], where the goal is to pack seven special tiles into a given pattern specified only by its boundary. There are several such silhouette puzzles [10], including the fourteen-tile "loculus" of Archimedes and the seven-tile "Sei Shonagon Chie no Ita" from Japan. Two tile sets and example arrangements are shown in Figure 1.* The appeal is universal, and there are rich opportunities for exploration [3, 4] and pedagogical use [12]. So it is not surprising that Tangrams have attracted the attention of mathematicians: Scott [8] and Wang and Hsiung [11] proved that exactly 13 convex polygons can be formed with the seven Tangram tiles. Read [7] showed that 4,842,205 Tangram patterns exist that are *snug*, meaning that the pattern has no holes and the tiles abut each other along an entire edge (or half edge in the case of the large triangles). He also showed that the number increases to 5,583,516 if holes are allowed. Graber et al.[5] showed that 5,520 "star" Tangrams exist, where all seven pieces meet at one point.

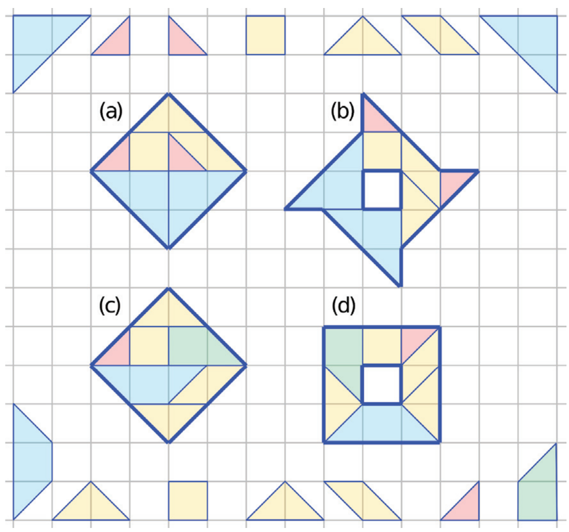

Figure 1 Tangram tiles along top, Sei Shonagon Chie no Ita tiles along bottom, all color-coded by area, and two example patterns that can be formed by each. These patterns all have *four-fold rotational symmetry*: rotation by ninety degrees leaves the boundary unchanged. Here, the boundaries are indicated by thick line segments. Patterns (a,c,d) also have *reflection symmetry*: reflection in a mirror leaves the boundary unchanged. Pattern (b) is a *chiral* pinwheel: reflection in a mirror changes the boundary between left- and right-handed versions that cannot be superimposed by rotation.

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*The online version of this article features color diagrams that might be easier to read.

The arguments presented by Read [7] and Wang and Hsiung [11] rely on the fact that Tangram tiles can be dissected into congruent right isosceles triangles joined along equal-length edges; such shapes are now called *polyabolos*. As can be seen in Figure 1 from the superposition of the patterns onto a square grid, the Tangram and Sei Shonagon Chie no Ita tile sets can both be dissected into 16 base units. Observe there that the seven Tangram tiles consist of two size-1 base unit triangles (shown in red), three different size-2 polyabolos (yellow), and two size-4 triangles (light blue). In contrast, the Sei Shonagon Chie no Ita tiles consist of one base unit (red), four size-2 polyabolos (yellow), one size-3 polyabolo (green), and one size-4 polyabolo (blue). If such tile sets are arranged with snug abutments, then the pattern is a size-16 polyabolo.

Wang and Hsiung showed that there exist exactly twenty size-16 convex polyabolos, thirteen of which can be constructed using Tangram tiles. More recently Fox-Epstein et al. [2] showed that Sei Shonagon Chie no Ita tiles can form sixteen convex polygons, and that exactly four seven-piece dissections of the square exist that can form nineteen out of the twenty size-16 convex polyabolos. Integer sequence A006074 [9] gives the number $a(n)$ of distinct size-n polyabolos as

$$\{1, 3, 4, 14, 30, 107, 318, \ldots\},$$

not counting reflections as distinct. Since $a(2) = 3$, Tangrams use all three of the size-2 polyabolos. Also, since $a(4) = 14$, the fourteen size-4 polyabolos shown in Figure 2 cover all distinct possibilities, not counting reflections as unique; this result is needed below. The value of $a(16)$ is not yet known, but it extrapolates exponentially versus n to roughly thirty million. Read's result implies that about 1/5 of these can be formed by Tangrams.

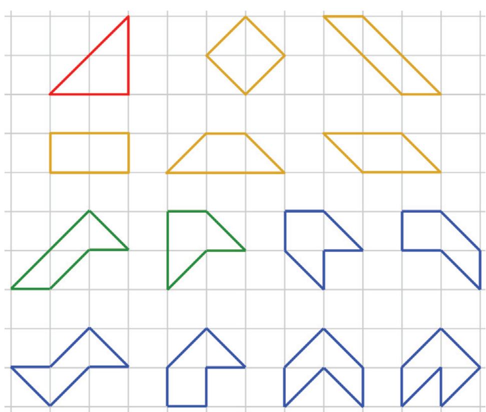

Figure 2 The fourteen size-4 polyabolos, color-coded by the number of edges.

While convexity and snugness are interesting properties for categorizing Tangrams, another aesthetic way is by their symmetry under reflection or rotation. For example those in Figure 1 all have *four-fold rotational symmetry*, meaning that rotation by multiples of ninety degrees leaves the boundary unchanged. It is natural to wonder: Do any other such four-fold Tangrams exist? In fact, there are only two. This is proved, below, using an approach similar in spirit to Wang and Hsiung [11]. I first establish some lemmas enabling the construction of a small number of candidate patterns, and then I test which of these may be formed from the seven Tangram tiles.

Construction of four-fold Tangrams

To begin, it is conceivable that some four-fold Tangram patterns consist of disconnected tiles that do not touch, or touch only at points. One such possibility is four disconnected congruent size-4 shapes, arrayed at multiples of ninety degrees with respect to one another. Since there are two size-4 triangular tiles, which may not be physically cut, the remaining five tiles would be required to form two additional size-4 triangles. This cannot be done. Other disconnected possibilities are a central four-fold shape of size $C = 4, 8$, or 12, surrounded by four congruent shapes of size $S = 3, 2$, or 1 respectively (subject to $C + 4S = 16$). Case $C = 4$ fails because a size-4 triangular tile can be neither the central shape (it is not four-fold) nor a surrounding shape (it is too large). Case $C = 8$ requires that the two triangular tiles form a central square, but fails because the remaining five tiles cannot form four congruent size-2 shapes. Case $C = 12$ fails because there are only two size-1 triangular tiles. All possibilities with disconnected shapes are thus eliminated. This establishes Lemma 1:

Lemma 1. *All four-fold Tangrams are simply- or multiply-connected planar solids.*

Lemma 1 implies that if a four-fold Tangram is literally dissected into four congruent shapes, then at least one tile must be physically cut. Otherwise, intact groupings of tiles could be slid apart into four separated congruent shapes. A physical cut can be made along an edge or diagonal of the square grid underlying the tile, so that it is split into two polyabolos. The corresponding cuts in the other quadrants of the dissection are mutually oriented at ninety degrees and must also split tiles into polyabolos or be along snug abutments of tiles. Therefore, tiles in a quadrant must solidly span and snugly abut two bounding cuts oriented at ninety degrees. If less than three tiles are involved, as happens in at least one quadrant since there are only seven tiles, then they must be aligned on the same square grid. Such alignment is required by symmetry in the other quadrants. This establishes Lemma 2:

Lemma 2. *All four-fold Tangrams are size-*16 *polyabolos.*

Since a four-fold Tangram is a size-16 polyabolo, it may be dissected into four congruent size-4 polyabolos each rotated ninety degrees with respect to two snugly abutting neighbors. As noted above, there are exactly fourteen distinct size-4 polyabolos, all shown in Figure 2. Therefore, all four-fold size-16 polyabolos may be constructed by testing all the ways to snugly assemble four copies of each of the Figure 2 shapes into a four-fold polyabolo. It is not onerous to try all possibilities and thereby establish Lemma 3:

Lemma 3. *There exist exactly ten four-fold size-16 polyabolos (not counting reflections as distinct), as depicted in Fig. 3.*

In principle these could have been mined from the thirty million or so $n = 16$ polyabolos, were they known.

The final step is to play the Tangram game of trying to pack all seven tiles into Figure 3 silhouettes. Possible arrangements can be systematically tested by (1) placing the first large triangle into all possible locations on the underlying grid, (2) for each of these, placing the second large triangle into all available locations on the underlying grid, (3) for each of these, placing the square into all available locations on the underlying grid, and so-on for the remaining tiles. We thereby establish the main proposition:

Proposition 1. *There exist exactly two four-fold Tangrams (not counting reflections as distinct).*

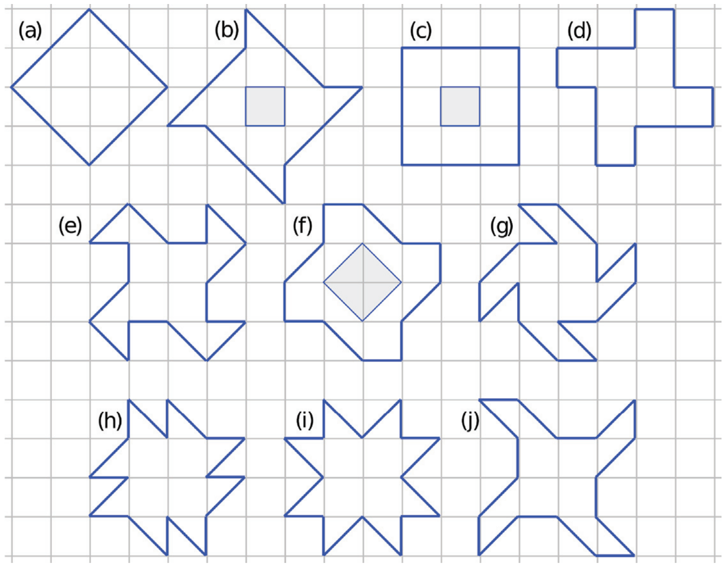

Figure 3 The ten size-16 polyabolos with four-fold rotational symmetry. Only three have reflection symmetry. The other seven are chiral, with left- and right-handed versions interchanged by reflection.

The first of these is the iconic simply-connected square of Figure 1a. The second is the pinwheel with square hole of Figure 1b. This pinwheel does not appear in any publication I have examined, of which Elffers and Schuyt [1], Read [6], and Slocum [10] are particularly comprehensive and scholarly, but it can be found on the internet.

With the same approach, we find that Sei Shonagon Chie no Ita tiles can form exactly two four-fold patterns (Figures 1c, d), both well-known. Thus, in both Tangram and Sei Shonagon Chie no Ita the classical square is the only four-fold and simply-connected pattern using all pieces. Similarly, we find that the Fox-Epstein, Katsumata, and Uehara dissections {a, b, c, d} can respectively make {3, 1, 3, 2} four-fold polyabolos: All can form the solid square, two can form the square with a square hole, two can form the top-right pattern in Figure 3, and one can form the pinwheel in Figure 1b, which thus deserves to be better known as an iconic Tangram pattern. These dissections can actually form an infinite number of four-fold patterns that are not polyabolos, since each set of tiles can form four disconnected size-4 parallelograms.

Other four-fold polyabolos

Four-fold polyabolos of different sizes may be constructed, per above, by four-fold assembly of known polyabolos. The number $a(n)$ of existing four-fold polyabolos of size-n is $a(n) = 0$ for n odd, of course. We find

$$a(n) = \{1, 1, 1, 1, 3, 4, 6, 10, 18\}$$

for

$$n = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}.$$

Interested readers may verify (or extend!) this sequence. Some of these patterns for $n < 16$ can be formed with a subset of the seven Tangram tiles; all such are shown in Figure 4. Traditional Tangram puzzles require the use of all seven tiles. This is satisfied by exactly three *sets* of Figure 4 patterns: {a,c,d}, {a,i}, and {d,d}. These could be added to the cannon of Tangram puzzles. Members of such sets could be stacked on top of one another to form additional (albeit highly improper!) four-fold Tangrams. Similarly, four sets of Figure 4 patterns can be formed using all the Sei Shonagon Chie no Ita tiles: {a,h}, {a,j}, {b,f}, and {b,g}. Interested readers may similarly explore the sets of four-fold patterns that can be formed by the Fox-Epstein, Katsumata, and Uehara tiles.

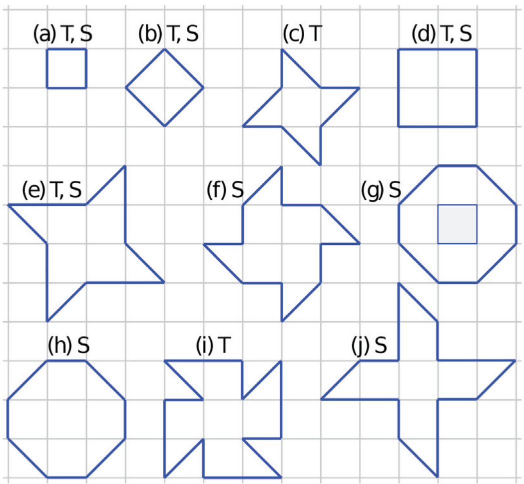

Figure 4 Four-fold polyabolos that can be formed by a subset of Tangram (T) and Sei Shonagon Chie no Ita (S) tiles. The shapes in the top row have size 2, 4, 6, and 8 left-to-right; the shapes in the middle and bottom rows have size 12 and 14 respectively.

Extensions

In closing, I would like to pose some further questions along the above lines. How many size-16 polyabolos exist with other kinds of symmetry, and how many of these can be formed by Tangram and Sei Shonagon Chie no Ita tiles? This would include patterns with one or two axes of reflection symmetry, and patterns with two-fold rotational symmetry. What alternative seven-piece dissections are optimal for constructing each category of pattern? How do the fractions of symmetric and constructible patterns vary with the number of base units in the "home" square (the possible square polyabolo sizes are two and four times an integer squared) and the number of tiles in the dissection? The Chinese Tangram is the most popular and well-known silhouette puzzle, but is this merely historical accident or is there some sense in which it represents an optimal dissection of the square?

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Summary. It is proved by construction that only two Tangrams exist with four-fold rotational symmetry. One is the iconic simply-connected square, and the other is a relatively unknown chiral pinwheel with a central square hole. Both are size-16 polyabolos.

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