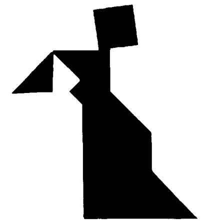
C H A P T E R

THREE



Tangrams, Part 1

"The seven Books of Tan . . . illustrate the creation of the world and the origin of species upon a plan which out-Darwins Darwin, the progress of the human race being traced through seven stages . . . up to a mysterious spiritual state which is too lunatic for serious consideration."

—Sam Loyd, *The Eighth Book of Tan*

One of the oldest branches of recreational mathematics has to do with dissection puzzles. Plane or solid figures are cut into various pieces, and the problem is to fit the pieces together to make the original figure or some other figure. The outstanding recreation of this type since the Renaissance is the Chinese puzzle game known as tangrams.

Although tangrams and jigsaw puzzles have a superficial resemblance, they are poles apart in the kinds of challenge they offer. As Ronald C. Read points out in his book *Tangrams: 330 Puzzles,* a typical jigsaw puzzle consists of hundreds of irregularly shaped pieces that fit together in just one way to make a large pattern. Little skill is required, just time and patience. Tangrams have only seven pieces, called tans. They are of the simplest possible shapes and are used to make an infinite variety of tangrams. In creating these figures a heavy demand is made on one's geometrical intuition and artistic ability.

The tans are obtained by slicing a square to produce two large triangles, a middle-size triangle, two small triangles, a square, and a rhomboid (*see* Figure 9). Note that all the corners are multiples of 45 degrees. If a side of the square tan is taken as unity, a side of any tan has one of four lengths: 1, 2, $\sqrt{2}$, and $2\sqrt{2}$.

"At first we are amazed at the unfitness of the shapes . . . with which we are expected to accomplish so much," wrote Loyd, the American puzzle expert. "The number 7 is an obstinate prime which cannot be divided into symmetrical halves, and the geometrical forms . . . with harsh angles, pre-

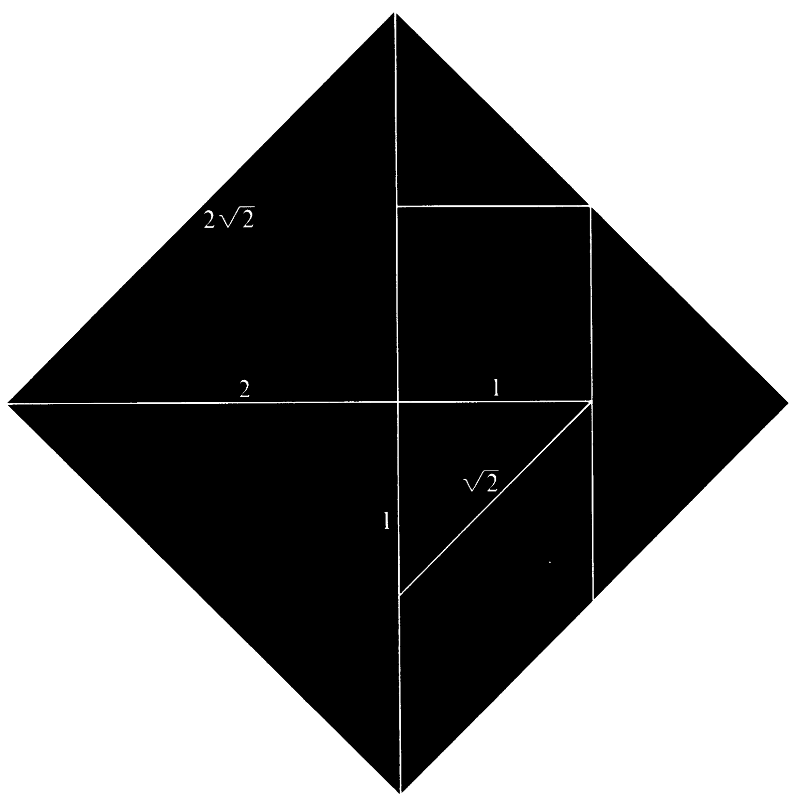


Figure 9 The seven tans

clude the possibility of variety, curves or graceful lines." After working for a while with the tans, however, one begins to appreciate the subtle elegance of the dissection and the richness of its combinatorial possibilities. All kinds of variant dissections, in imitation of tangrams, have been marketed from time to time, but not one has come even close to tangrams in popularity. As with origami, it is the very simplicity of the material and its apparent unfitness for artistic use that lie at the heart of its charm.

Tangram play falls roughly into three major categories:

1. Searching for one or more ways to form a given tangram, or for an elegant proof of the impossibility of forming a tangram.
2. Finding ways to depict, with maximum artistry or humor, or both, silhouettes of animals, human figures, and other recognizable objects.
3. Solving a variety of problems in combinatorial geometry that are posed by the seven tans.

Many books, and even a few encyclopedias, declare that tangram play is about 4000 years old. In my *Scientific American* column for September 1959, I called tangrams the oldest of dissection games and said that the Chinese had been amusing themselves with it for several thousand years. This is totally wrong. The man responsible for the myth is none other than Sam Loyd. In 1903, when Loyd was sixty-one and at the height of his fame, he published a little book (now extremely rare) called *The Eighth Book of Tan.* No Western book on tangrams has been more original or influential. In addition to containing hundreds of excellent new figures, Loyd invented a preposterous legend about the pastime's origin. It was the greatest hoax in the history of puzzledom, and the number of intelligent people taken in by it rivals the number of scholars who accepted H. L. Mencken's spurious history of the bathtub.

"According to the late Professor Challenor," Loyd wrote, "whose posthumous papers have come into the possession of the writer, seven books of Tangrams, containing one thousand designs each, are known to have been compiled in China over 4000 years ago. These books are so rare that Professor Challenor says that during a forty years' residence in China he only succeeded in seeing perfect editions of the first and seventh volumes, with stray fragments of the second.

"In this connection it may be mentioned that portions of one of the books, printed in gold leaf upon parchment, were found in Pekin by an English soldier who sold it for £300 to a collector of Chinese antiquities, who kindly furnished some of the choicest designs presented in this work."

According to Loyd, Tan was a legendary Chinese writer who was worshiped as a deity. He arranged the patterns in his seven books to display seven stages in the evolution of the earth. His tangrams begin with symbolic representations of chaos and the yin-yang principle. These are followed by primitive forms of life, then the figures proceed up the evolutionary tree through fish, birds, and animals to the human race. Scattered along the way are tangrams of human artifacts such as tools, furniture, clothing, and architecture. Loyd inserts remarks by Confucius, a philosopher called Choofootze, a commentator named Li Hung Chang, and his mythical Professor Challenor. Chang is quoted as saying that he knew all the figures in the seven books of Tan before he could talk. And there are references to a "well-known" Chinese saying about "the fool who would write the eighth book of Tan."

All of this, of course, was sheer fabrication. When Henry Ernest Dudeney, Loyd's British counterpart, wrote an article on tangrams for *The Strand Magazine* (November 1908), he soberly repeated Loyd's legendary history. This aroused the curiosity of Sir James Murray, the distinguished lexicographer and an editor of the *Oxford English Dictionary,* who made inquiries through one of his sons, then teaching at a Chinese university. Oriental scholars had never heard of Tan or even the word tangrams. The game, Murray informed Dudeney, is known in China as *ch'i ch'iao t'u,* meaning "seven-ingenious plan" or, less literally, "clever puzzle of seven pieces."

Murray could find no record of the word tangram earlier than in an 1864 Webster's dictionary. It had been coined about 1850, Murray guessed, by an American who probably combined *tang,* a Cantonese word for "Chinese," with the familiar suffix *-gram*, as in anagram or cryptogram. A different theory about the name has recently been advanced by Peter Van Note in his introduction to a Dover reprint of Loyd's fanciful book. Chinese families who live on riverboats are called *tanka,* and *tan* is a Chinese word for prostitute. American sailors, taught the puzzle by tanka girls, may have called it tangrams—the puzzle of the prostitutes.

When Dudeney reported Murray's opinions, in *Amusements in Mathematics* (pp. 43–46), he may have deliberately added a hoax of his own. An American correspondent, Dudeney writes, had told him that he owned a Chinese set of mother-of-pearl tans with an accompanying rice-paper booklet of more than 300 figures. The correspondent was puzzled by a mysterious inscription on the front page that he said he had tried to have translated, but no Chinese to whom he had shown it had been willing or able to read it. Dudeney reproduced the inscription and asked the reader for help. We do not know what the response was to this request, but Read, who owns a copy of the same booklet, had no difficulty clearing up the mystery. The inscription is nothing more than a caption under the tangrams of two men. The caption reads: "Two men facing each other and drinking. This shows the versatility of the seven-piece puzzle."

No one knows when tangrams originated. The earliest reference known is a book published in China in 1803. Its title, *The Collected Volume of Patterns of the Seven-Piece Puzzle,* suggests earlier books. Most scholars believe the game originated in China about 1800, became an Oriental craze, and then spread rapidly to the West. The earliest Western books, says Read, were little more than copies of Chinese rice-paper booklets. The Western books even copied errors in the Chinese illustrations.

One of the earliest English books on tangrams, originally owned by Charles Lutwidge Dodgson (better known as Lewis Carroll), came into Dudeney's possession. It is called *The Fashionable Chinese Puzzle* and was first published in New York in 1817. Dudeney quotes from it a passage stating that the game was a favorite of "ex-Emperor Napoleon, who, being now in a debilitated state, and living very retired, passes many hours a day in thus exercising his patience and ingenuity." This too is an unsupported statement, undoubtedly false. The puzzle is said by Loyd to have been a favorite of John Quincy Adams's and Gustave Doré's, although I know of no basis for either assertion. We do know that Edgar Allan Poe enjoyed the game, because his imported set of carved ivory tans is owned by the New York Public Library. An anonymous French work, *Recueil des Plus Jolie Jeux de Société* (1818), may be a translation of Dodgson's English book, or vice versa. I have not seen a copy of either. An 1817 American book bears the title *Chinese Philosophical and Mathematical Trangram.* "Trangam" was an old English word for a trinket, toy, or puzzle. Samuel Johnson misspelled it as "trangram," and the spelling persisted in later dictionaries. Did the book's anonymous author revive an obsolete word that later evolved to "tangram," or did he misspell "tangram," a word already in use? One mystery novel, *The Chinese Nail Murders* by the Dutch diplomat and Orientalist Robert Van Gulik, is woven around a set of tangram patterns.

Poe's tans are shown in Figure 10. The delicate filigree carving is characteristic of the old Chinese ivory tans. Note that the pieces pack into a square box in two layers. The two layers are squares of equal size, so that putting away the tans is a puzzle in itself. In nineteenth-century China, where tangrams were popular among adults (it is considered a child's pastime in the Far East today), the pieces were made in many sizes and from many different materials. Dishes, lacquer boxes, and even small tables were given the shapes of the tans.

So much for the historical background. Let us turn now to the first of the three categories of tangram play: solving given figures. Figure 11 shows a dozen interesting shapes on which the reader is invited to try his skill. Each

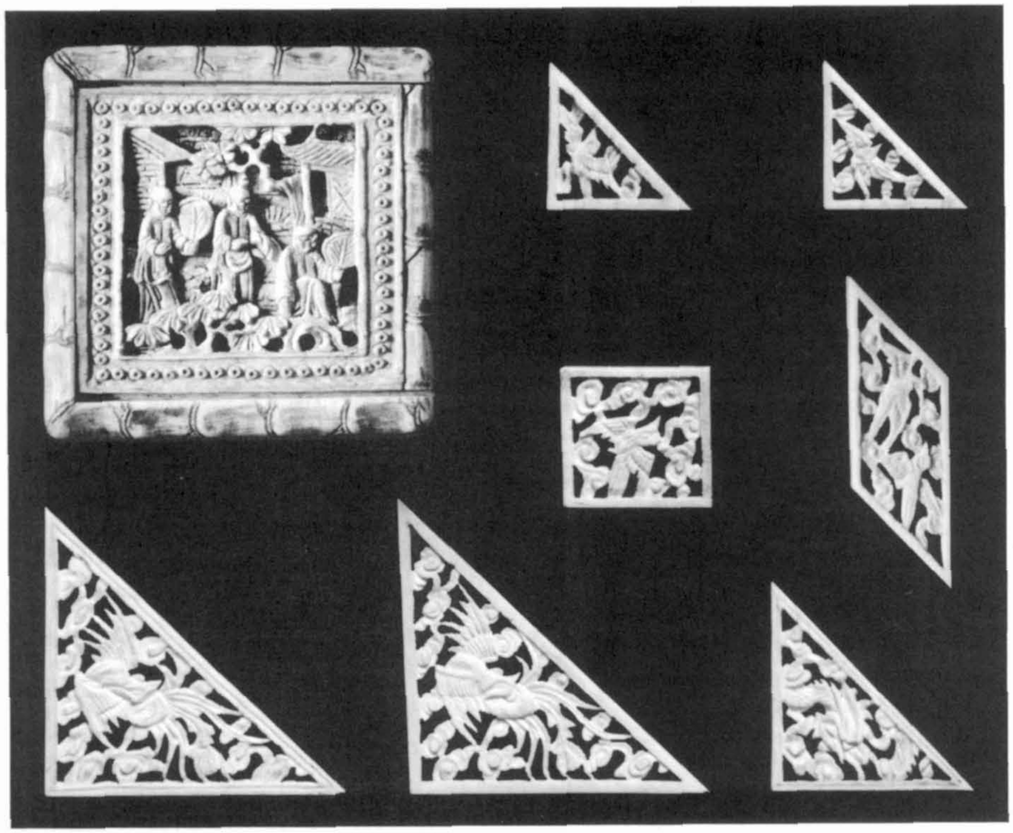


Figure 10 Edgar Allan Poe's carved ivory tangram set

requires all seven pieces. The rhomboid, the only asymmetrical tan, may be placed either side up. One figure in the illustration is not possible. Can the reader identify it and prove its impossibility?

The paired tangrams in Figure 12 are samples of delightful paradoxes introduced by Loyd. (The first three pairs were devised by Loyd, the fourth pair was devised by Dudeney.) Although the figure at the right in each case seems to be exactly the same as its mate, except for a missing portion, each is made with all seven tans!

The tangrams in Figure 13 are not intended as patterns to be solved, but as illustrations of the second category of play: creating artistic and amusing pictures. (I confess responsibility for the Nixon caricature.) "One remarkable thing about . . . Tangram pictures," wrote Dudeney, "is that they suggest

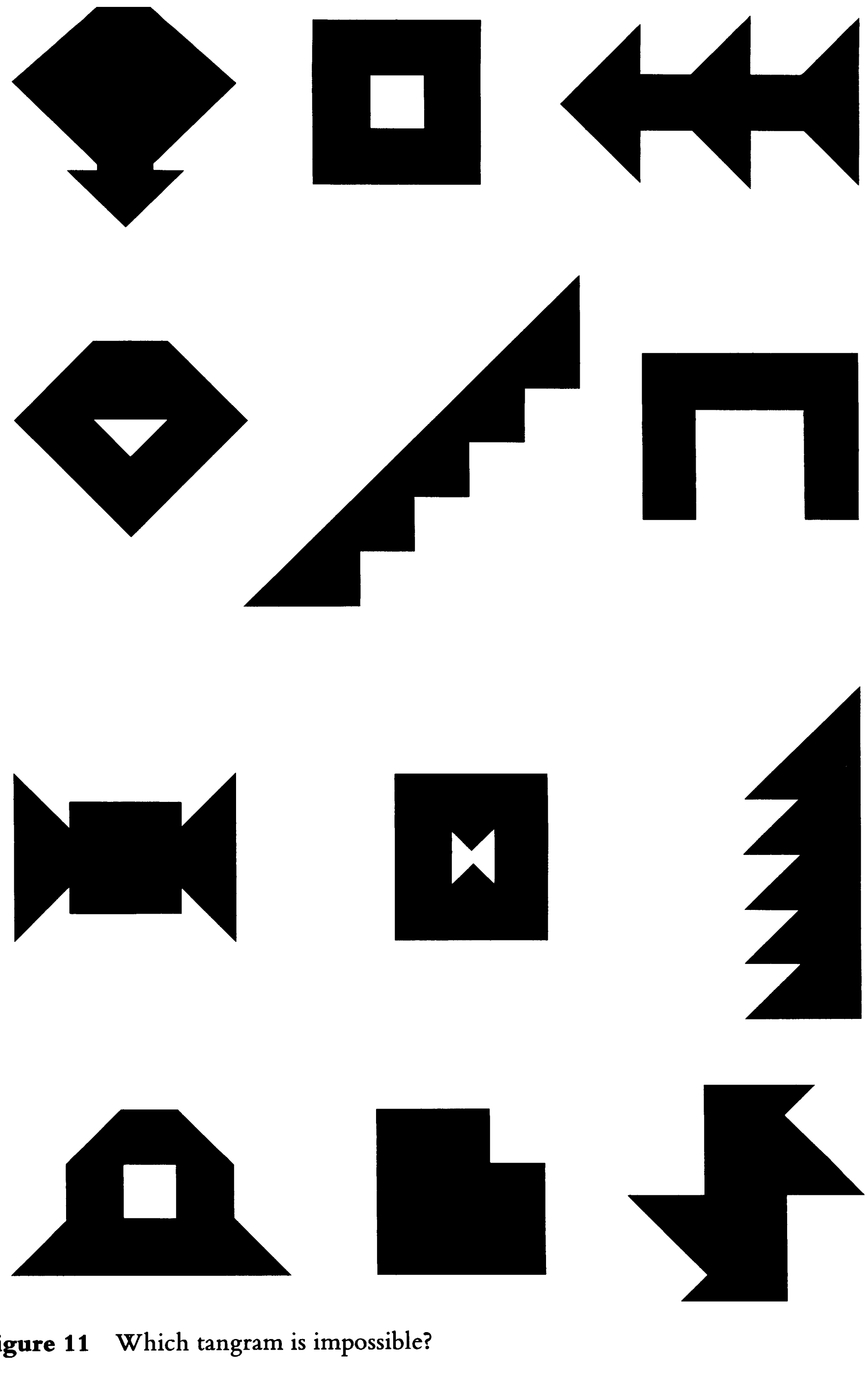


Figure 11 Which tangram is impossible?

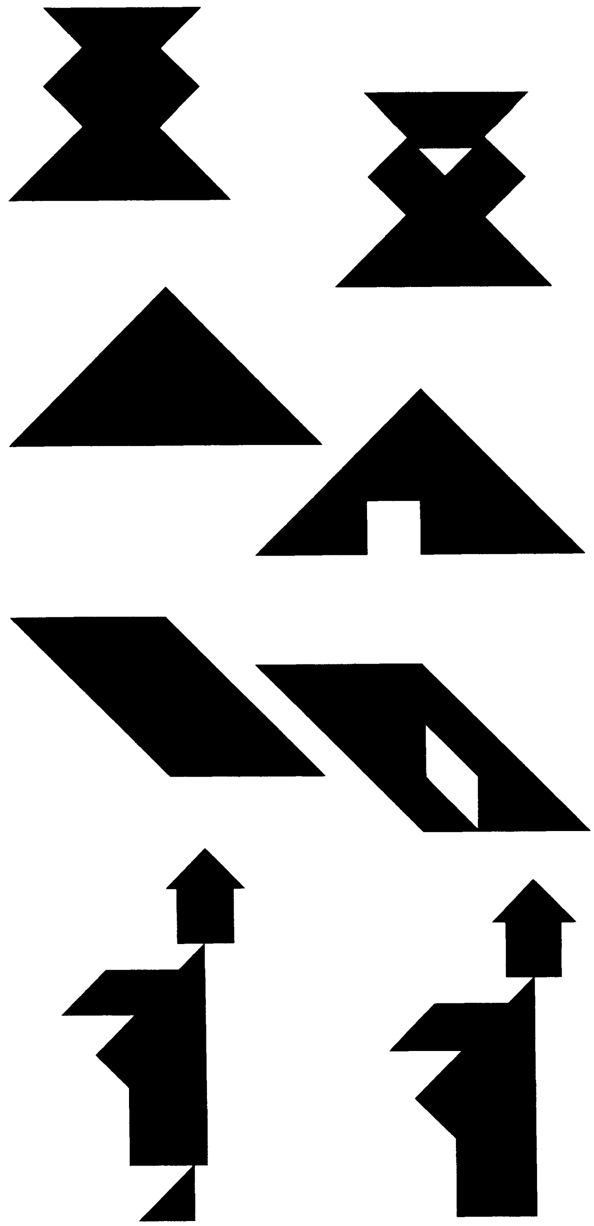


Figure 12 Tangram paradoxes

to the imagination such a lot that is not really there. Who, for example, can look . . . at Lady Belinda . . . without soon feeling the haughty expression . . . ? Then look again at the stork, and see how it is suggested to the mind that the leg is actually much more slender than any one of the pieces employed. It is really an optical illusion. Again, notice in the case of the yacht how, by leaving that little angular point at the top, a complete mast is suggested. If you place your tangrams together on white paper so that they do not quite touch one another, in some cases the effect is improved by the white lines; in other cases it is almost destroyed."

One can mix two or more sets of tans to produce more elaborate figures. Dudeney gives a number of these "double tangrams" in *536 Puzzles and Curious Problems* (pp. 221–222), and others will be found in Read's book. I agree with Read, however, when he writes: "With fourteen pieces to play around with, one cannot help but feel that it should be possible to arrive at a reasonable likeness of just about anything. Consequently, the sense of achievement that one gets on producing a recognizable cow, sailing boat, human figure, or what have you, from a mere seven pieces, is quite lacking."

Combining two related tangrams, each made with seven tans, is a different matter. Four classic examples, all devised by Loyd, are a woman pushing a baby carriage, a runner being tagged out at home plate, two Indian braves, and a man with a wheelbarrow (*see* Figure 14). Note that the man and the wheelbarrow are identical tangrams except for orientation.

The third category of tangram play, solving combinatorial problems, is the most interesting of all to mathematicians. There have been some remarkable

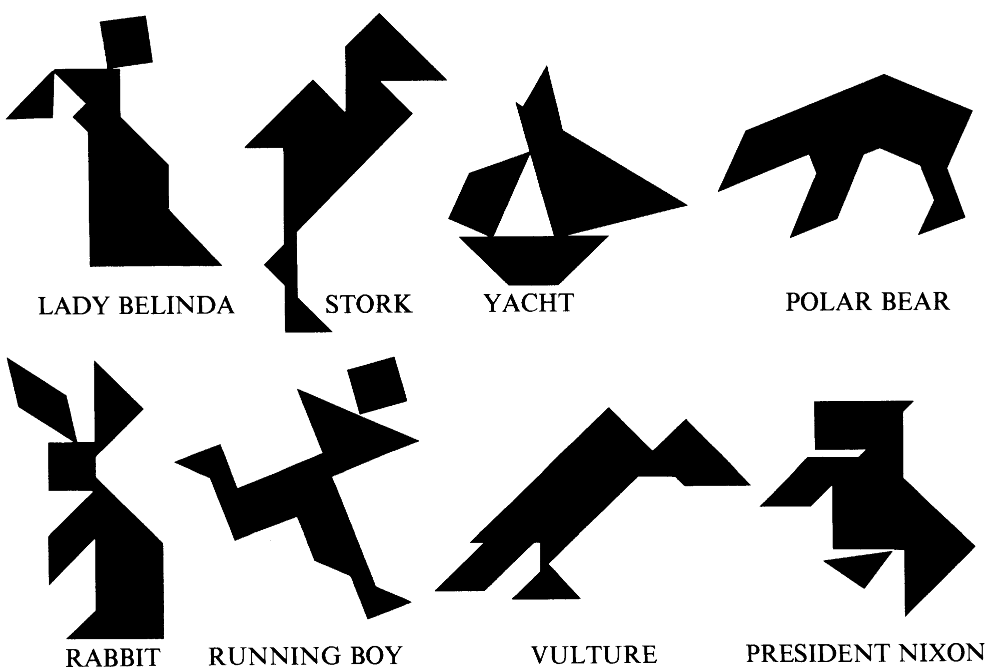


Figure 13 Tangram pictures

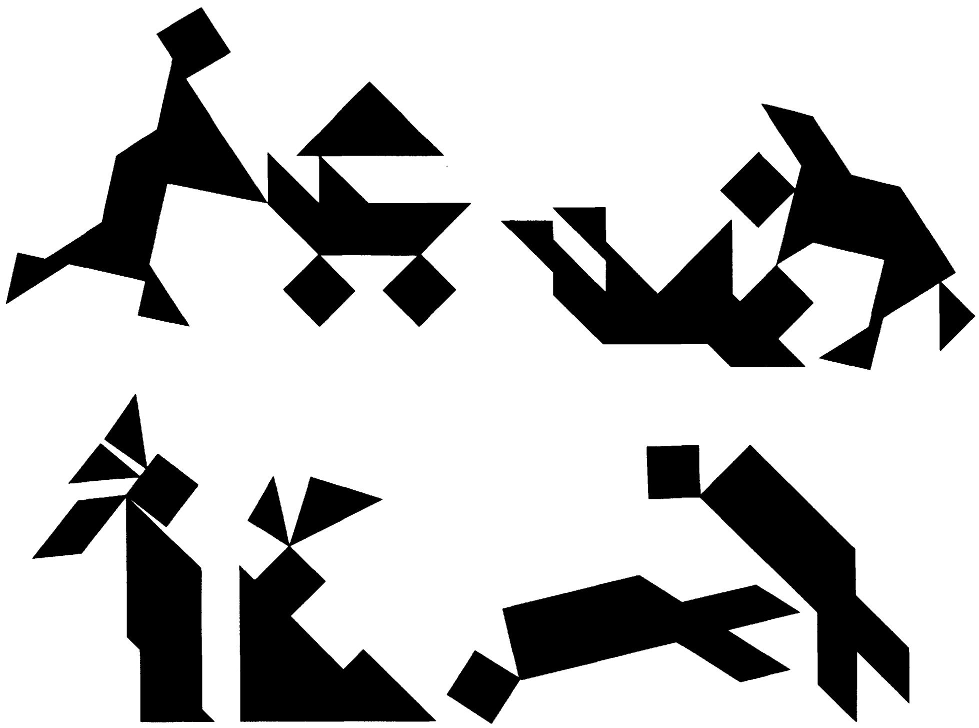


Figure 14 Double tangrams by Sam Loyd

contributions made to this field by Read, a specialist in graph theory at the University of Waterloo, and by E. S. Deutsch, a computer scientist with P. S. Ross and Partners in Toronto. Some of their results will be presented in the next chapter. To whet the reader's appetite, here are two problems that will be answered in the next chapter.

1. How many different convex polygons can be formed with the seven tans? There must not be any "windows" in the figures. Rotations and reflections are not, as is customary, considered to be different. Because all three-sided polygons are convex, and no nonconvex polygon of four sides can be made with all seven tans, answering this question also gives the number of three- and four-sided polygons. It is easy to see that only one triangle is possible (since corners must be multiples of 45 degrees, the triangle must be a right isosceles one), but finding all the higher convex polygons is a bit tricky.
2. How many different five-sided polygons can be made?

ANSWERS

The impossible tangram in Figure 11 is the square with the central square hole. The two large triangles can go only in opposite corners. The square tan must go in a third corner and the rhomboid must touch the fourth corner, but now there is no spot for the middle-size triangle.

B I B L I O G R A P H Y

The Bibliography following Chapter 4 (p. 54) is also for this chapter.

C H A P T E R

FOUR

Tangrams, Part 2

> "The formation of designs by means of seven pieces of wood . . . known as tans . . . is one of the oldest amusements in the East. Many hundreds of figures representing men, women, birds, beasts, fish, houses, boats, domestic objects, designs, etc., can be made, but the recreation is not mathematical, and I reluctantly content myself with a bare mention of it."
>
> —W. W. Rouse Ball, *Mathematical Recreations and Essays*

Not mathematical? Ball wrote without giving the matter much thought. In this chapter we consider some nontrivial combinatorial problems presented by the seven tans.

A question that arises at once is: How many sides can a tangram formed with all seven tans have? Although the answer is obvious, it seems to have been first

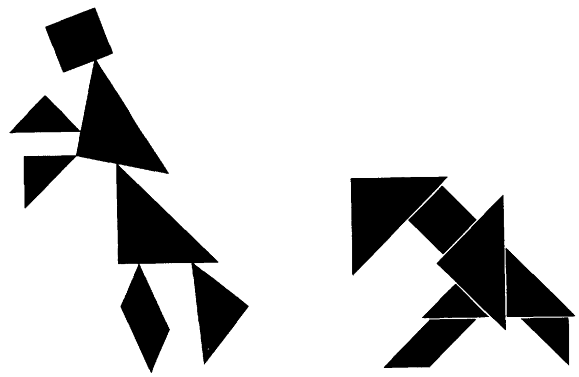


Figure 15 Improper tangram (left) and proper tangram (right), each with twenty-three sides

stated by Harry Lindgren in his 1968 article "Tangrams." This is how Ronald C. Read, a mathematician at the University of Waterloo, puts it in a lengthy communication to me from which I shall be quoting liberally:

"Tangrams have 23 sides between them, so that a tangram like that (at the left in Figure 15) will clearly have this number of sides. On the other hand, tangrams with pieces joined only at a corner are mathematically uninteresting . . . . " Let us make a rule, Read proposes, that a tangram must have a perimeter topologically equivalent to a circle, that is, it must not self-intersect. Read calls these "proper tangrams." How many sides can a proper tangram have? Again the answer is twenty-three. The proof is supplied by the figure of a bowing man shown at the right in Figure 15 and by almost endless other examples.

The proper tangrams contain an important subset that Read calls "snug tangrams." In order to understand the meaning of snug, draw lines on all tans (except the two small triangles) to create sixteen identical right-isosceles triangles with unit legs (*see* Figure 16). A snug tangram is a proper tangram formed so that, where two tans are in contact, the sides of the small right triangles match exactly, either leg to leg or hypotenuse to hypotenuse. All convex tangrams are snug, and so are many of the traditional figures (*see* Figure 17).

Snugness, by the way, is characteristic of technology in the Orient, where the dimensions of houses, furniture, and so on tend to be exact multiples of a basic length. The Japanese building industry, I am told, is one of the most efficient in the world because Japanese lumber is standardized in lengths that are multiples of a basic "mat" length.

In addition to snugness of fit, Read adds two more limitations: A snug tangram must be simply connected (all in one piece), and there must be no

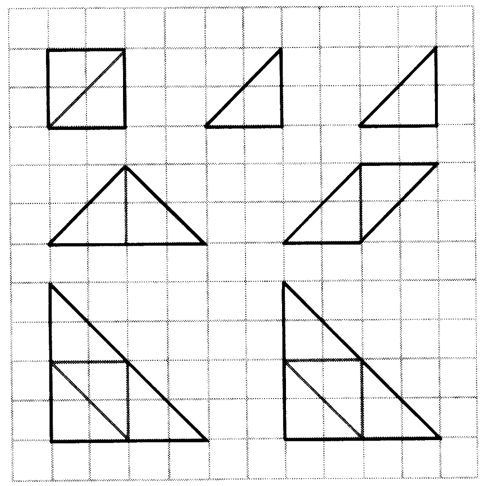


Figure 16 The seven tans

interior holes, including holes that touch the perimeter at one or more single points. It is convenient to diagram snug tangrams on graph paper so that all integral edges are on the orthogonals of the matrix. All diagonal edges will then be multiples of $\sqrt{2}$ and therefore irrational. This suggested to Read a very pretty problem: How many snug tangrams have *all* sides irrational? Such tangrams, if diagrammed on graph paper, would have every side running diagonally.

The tans between them have a total of thirty side segments, Read continues, but "whenever we place two pieces together, the two sides that abut are lost to the perimeter, and it can happen that we lose more than two. Furthermore, in order that the resulting tangram shall be connected, there must be at least six lines along which two pieces come together. Hence we cannot avoid losing 12 segments, therefore the total number of segments on the outside cannot be

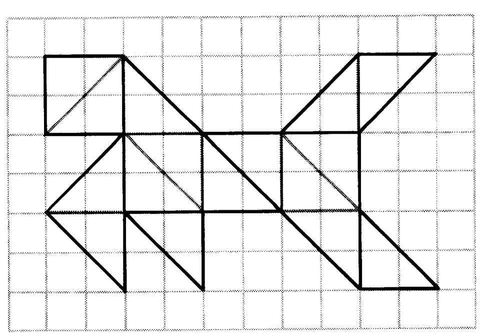


Figure 17 A snug tangram of a dog with eighteen sides

more than 18. Each side of a snug tangram consists of at least one segment, hence the total number of sides cannot exceed 18 either." The dog in Figure 17 proves that a snug tangram can have this maximum total of sides.

The number of proper tangrams obviously is infinite. You have only to observe that two tans can abut in an infinite number of positions. If the question is confined to certain kinds of tangram, however, interesting problems in combinatorial enumeration arise. For example, how many convex tangrams are there? A convex tangram is a polygon in which all corner angles are less than 180 degrees. That there are only thirteen was proved in 1942 by Fu Tsiang Wang and Chuan-Chih Hsiung in "A Theorem on the Tangram" in *The American Mathematical Monthly*. The thirteen are shown in Figure 18. If mirror images count as different convex tangrams, then there are eighteen. The eighteen convex tangrams appear in Chinese tangram books with solutions showing that all can be made without turning over the asymmetric rhomboid tan. (Interior lines are omitted in the illustration, in case some readers might enjoy solving them.)

The thirteen convex tangrams include all three- and four-sided polygons that can be made with the seven pieces. As stated in the previous chapter, nonconvex quadrilaterals are not possible. (Can you prove this? Hint: The four interior angles of such a four-sided figure would have to be three angles of 45 degrees and one of 225 degrees, and the figure would have to consist of sixteen isosceles right triangles congruent with the small triangular tan.) Five-sided polygons, made with the tans, *can* be nonconvex. How many pentagons, both convex and nonconvex, Lindgren asked, can be made with the seven tans?

At this spot in my 1974 column, I gave a proof that there are sixteen snug pentagons and two "loose" pentagons, making eighteen in all. Unfortunately, there was a flaw in the proof, which readers quickly discovered. For months I was flooded with hundreds of letters from readers of all ages who had found pentagons not among the eighteen that I had given in an illustration.

I kept tab of the number of different pentagons sent by readers until it reached a maximum of twenty-two snug and thirty-one loose ones, or fifty-three in all. Only two readers, each working independently and by hand, found all fifty-three. They are Allan L. Sluizer of Northbrook, Illinois, and Åke Lindgren of Uppsala, Sweden. In 1975 Dr. I. Takeuchi at the Institute for Electrical Engineers, Musashimo, near Tokyo, confirmed the fifty-three figures by a computer program. In 1976, unaware of Takeuchi's program, Michael Beeler of Cambridge, Massachusetts, wrote a program that gave the same results. Beeler's drawings of the fifty-three pentagons are shown in Figure 19.

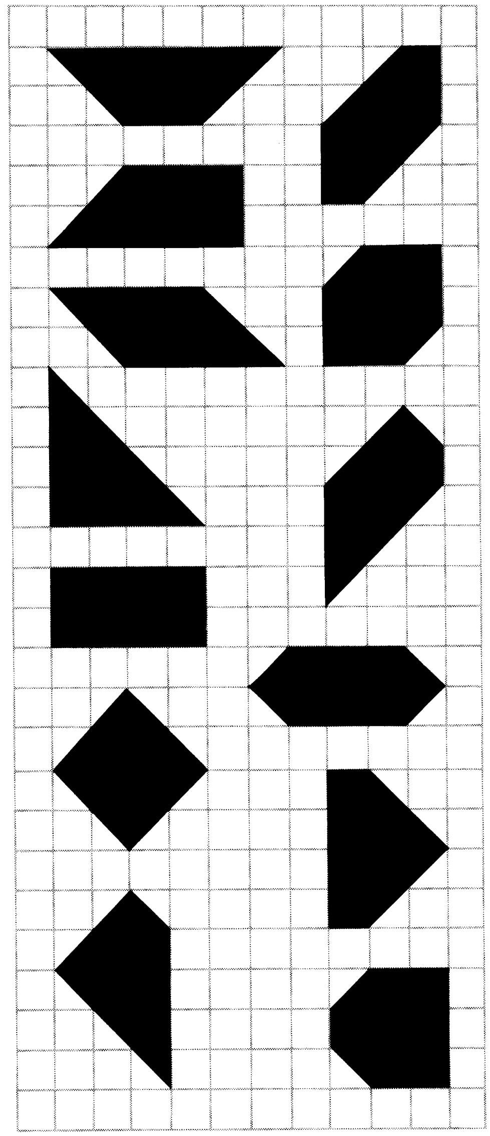
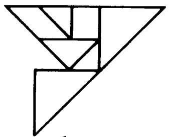


Figure 18 The thirteen convex tangrams

One might now wonder how many tangram polygons have six sides, or seven or more, but (as Read points out) the question is easily answered. For $n = 6$ through 23 there is an infinity of n-sided polygons. You have only to glance at pentagon 28 in Figure 19 to see that by sliding the large triangle on the left along the hypotenuse of the other large triangle, you can create an infinity of hexagons. How many snug hexagons are there? Although it is a finite number, as far as I know the number has not yet been determined.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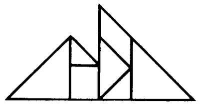
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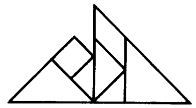
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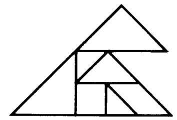
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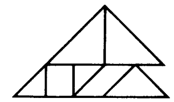
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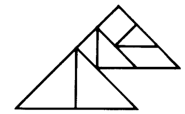
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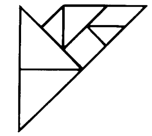
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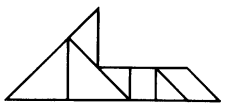
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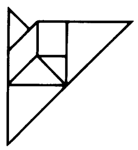
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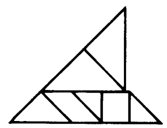
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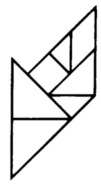
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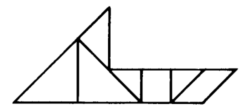
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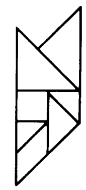
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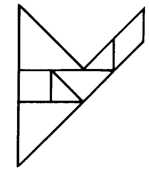
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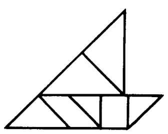
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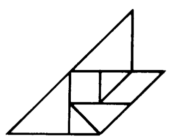
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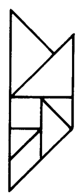
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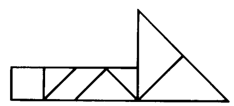
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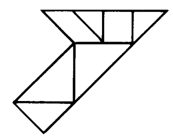
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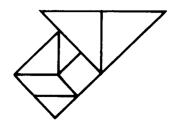
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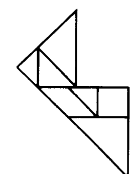
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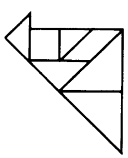
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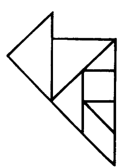
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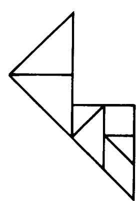
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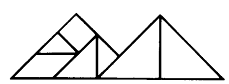
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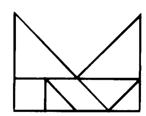
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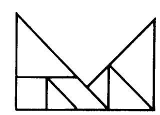
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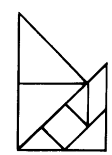
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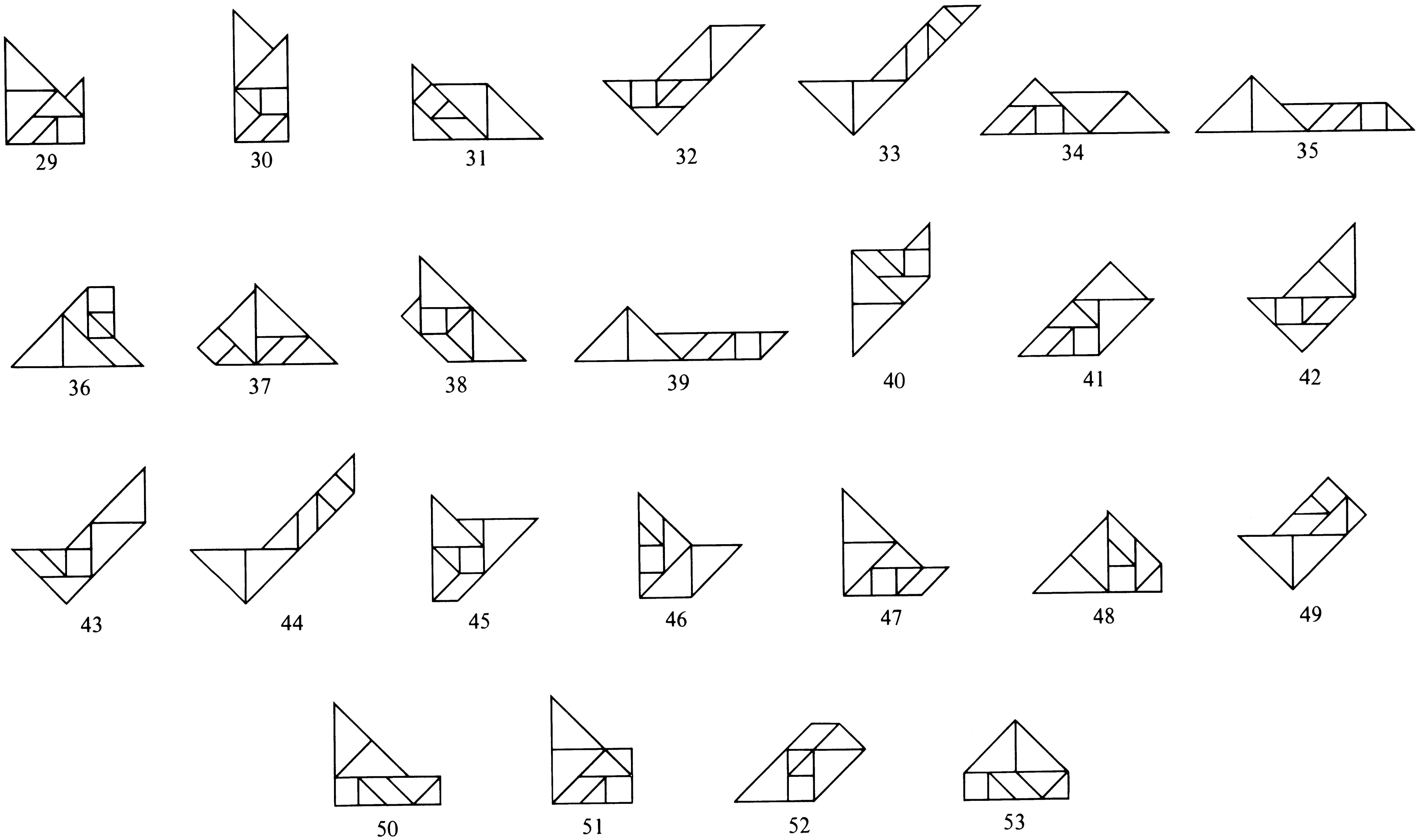

Figure 19 The fifty-three pentagons

There are, of course, only a finite number of snug tangrams, but the exact number (Read calls it the snug number) also is far from known. Read has devised an ingenious procedure by which a computer could be programmed to count the number, but he estimates that it is well into the millions, and no such program has yet been written. Unfortunately, the details of Read's procedure are too complex to give here. A simpler problem, using exactly the same procedure, has been solved, however. Read defines a minitangram as one made with the five pieces that remain after leaving out the two large triangles. The problem of finding all snug minitangrams is so much simpler than finding all snug tangrams that Read was able to write a computer program to solve it and ran it on, appropriately enough, a minicomputer at the University of Waterloo. It took only half an hour, and the count was 951. The computer was hooked up to a display terminal, so that it drew pictures of all the minitangrams.

Read's programs are designed only for enumeration, not for solving individual tangrams. Is it possible to write a program that will inspect any given tangram and search for at least one solution? Yes, such a program has been developed and published by E. S. Deutsch, a computer scientist. In theory it is possible to write a program that will systematically examine all possible ways the tans fit a given tangram and then print all the solutions, but the complexity of such a program is so great that no one has yet attempted it. Deutsch's program is not of this type. It is heuristic, which means that it goes about solving a tangram in much the same way a person does: by applying a series of tentative tests, examining the feedback, backtracking and trying again when no solution is found, and continuing until it either discovers a solution or gives up. The program seldom fails, taking an average time of about two seconds to solve a tangram.

The program starts by examining the tangram's perimeter, noting the edge lengths, and the angles at each corner. It then attempts to separate the tangram into two or more subtangrams. For instance, if two portions of the tangram meet at a point, each portion clearly is a separate tangram. If, say, a rabbit has two ears formed by the two small triangles, each ear meeting the head at a point, the program instantly identifies the two pieces, removes them and goes to work on the subtangram that remains. If the tangram does not have portions meeting at points, the program explores ways of dividing it into subtangrams by extending edges from a corner into the figure. In many cases the internal extension of an edge clearly divides the tangram into subtangrams; in other cases the extension is only a possible line of division.

After the program's preliminary exploration, it applies a series of heuristic tests until it finds a way of fitting tans into a subtangram or a possible subtangram. When a fit is found, the subtangram is extracted and the program turns to what remains. The tests are ranked in order of their efficacy, so that the strongest tests can be applied first, then the next strongest, and so on. If no solution is obtained, the program backtracks and begins with the second test. It is impossible to describe the program in more detail, but interested readers will find it fully explained, with flow charts and examples, in "A Heuristic Solution to the Tangram Puzzle," by E. S. Deutsch and Kenneth C. Hayes, Jr., in *Machine Intelligence* 7. A somewhat similar program was developed in 1972 by Ejvind Lynning, a Danish student working with Jacques Cohen, a physicist at Brandeis University.

Snug tangrams are, for obvious reasons, usually more difficult to solve (by a person or a computer) than nonsnug figures, and the difficulty tends to increase as the number of sides decreases. One might suppose a pattern with only one solution would be harder to solve than one with many, but that is not the case. A pattern in which the tans touch only at points has only one solution, but it is an immediately obvious one, and there are patterns with a large number of solutions that are among the most difficult.

The construction of tangrams with "holes" raises many curious new problems. It is not hard to form a square hole of area 4 or a triangular hole of area 2 that does not touch the border, or two triangular holes of areas 1 and $\frac{1}{2}$ that do not touch each other or the border (*see* Figure 20); ("touch" includes touching at a single point). Can the reader find a way to make exactly two holes, each 1×1 square, that do not touch each other or the perimeter? Or two holes, under the same provisos, one a triangle and one a square, each with an area of 1? They are not difficult tasks, but here are two more that I set for myself and found much harder: (1) Form just three holes, two triangular and one square, that do not touch one another or the border. (2) Form just three holes, two rectangular and one triangular, that do not touch one another or the border. Apparently it is not possible for three holes of this type to be all rectangular or all triangular, or for two triangular holes each to have an area of 1.

The "farm problem" is another unsolved hole problem. What is the largest hole not touching the border that can be inside a tangram? The solution is a limit that cannot be reached, but one can come as close to it as one wishes. (The best I can obtain is the limit 10.985+.) How many sides can a single hole that is simply connected and not touching the border have? The maximum surely is thirteen. What is the largest "farm" not touching the border that is square? Rectangular? Triangular?

Another unexplored type of tangram problem is finding ways of transforming one tangram into another with the fewest number of moves. A move

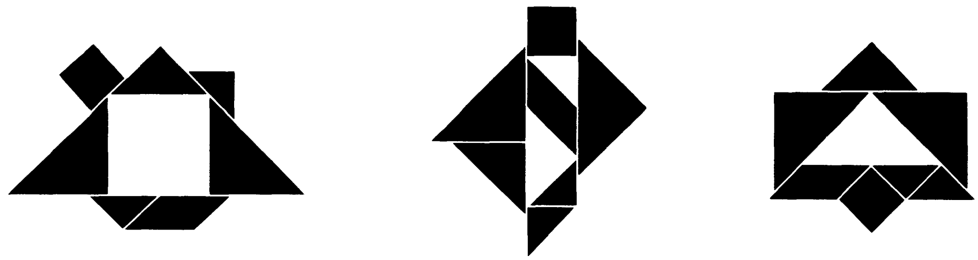

Figure 20 Tangrams with holes

consists in altering the position of a set of one or more tans without disturbing the pattern of the set. For example, the large square tangram can be changed to the large triangle or the large rhomboid in one move, or to the 2 × 4 rectangle in three moves. As Read notes in his book, the square can be changed to a 3 × 3 square with a missing 1 × 1 corner in just two moves.

Still another area of tangram play open to exploration is the devising of competitive games that use one or more sets of tans. The only tangram game I have seen in books in English is the party game of giving each guest a set of tans and awarding prizes to those who are the first to make each of a series of displayed patterns. Read's concept of snugness suggests a variety of two-person games. Here are three that occurred to me. In playing them it is a good plan to mark the middle of the long edges to facilitate placing the tans snugly.

1. Snuggle up. Begin with the tans forming the large triangle, the square, or any four-sided polygon. Players alternate moves. A move consists in changing the position of a single tan to form a new snug tangram that has more sides than the preceding one. The first player who is unable to make a move loses.
2. Snuggle down. Same as above, except that the initial tangram is an eighteen-sided snug tangram and each move must decrease the number of sides. As in the preceding game, you cannot move a piece that leaves or forms a hole or that divides the figure into parts that touch only at points. Both games end quickly. Because a snug tangram must have at least three sides and no more than eighteen, a game cannot last beyond fifteen moves. Games that go a full fifteen moves are possible.
3. Snuggle up and down. Start with a ten- or eleven-sided snug tangram. One player must increase the number of sides on each play, the other must decrease the number. The same piece may not be moved twice in succession. Each player keeps a record of his increases and decreases, and the first to score 30 wins. If a player cannot move, he loses. If the up

player makes an eighteen-sided tangram, he wins. If the down player makes a four- or three-sided tangram, he wins. This game lasts considerably longer than the other two, and it often takes unexpected turns. One player can get far ahead in scoring only to discover, just before he expects to make a winning move, that no move is available.

In all three games it is good to keep a running record of the number of sides because it is easy to forget the number, and time is lost by repeated counting. A cribbage board is a convenient device not only for recording the number of sides but also for keeping score in the up-and-down game.

ANSWERS

Solutions to the four hole problems are shown in Figure 21. At the top left in the illustration is a way of making two unit square holes. At the top right are two holes, one a unit square and the other a triangle of area 1. At the left on the bottom of the illustration is a way of making a square hole and two triangular

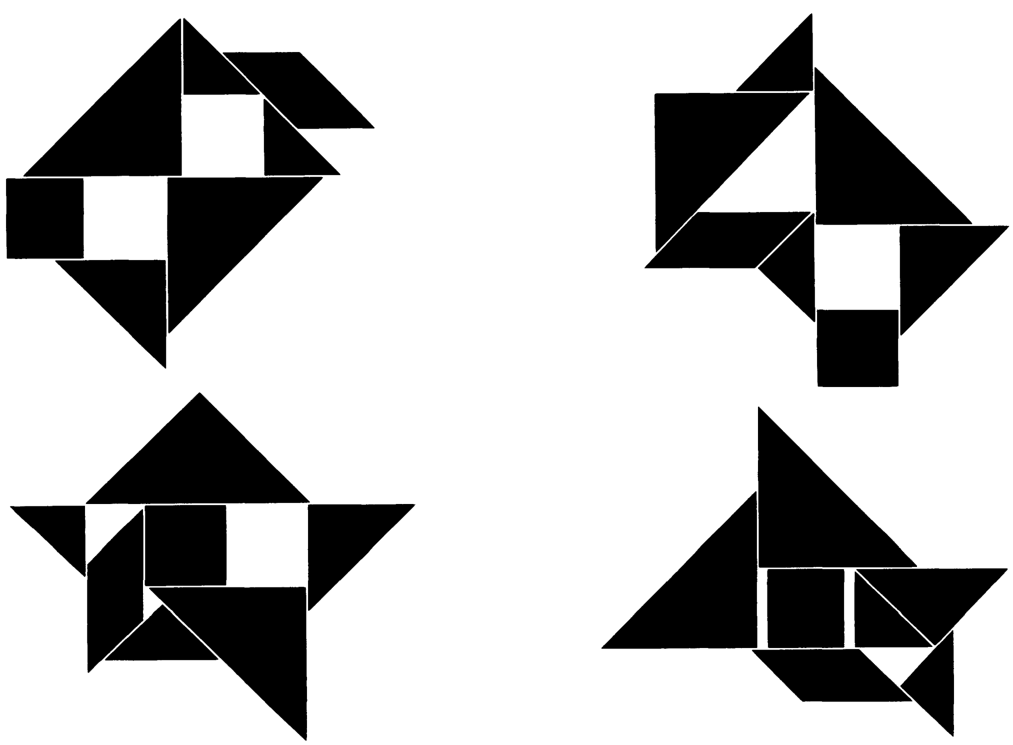

Figure 21 Solutions to hole problems

holes. (The fit is extremely close. The top triangle's hypotenuse is longer than the length it must span by only .121 + .) At the right on the bottom are two rectangular holes and one triangular hole.

Read has proved that a tangram that is snug except for one or more holes cannot have more than one hole if the holes do not touch one another or the border. The smallest possible hole is equal to a small triangular tan. No matter how two such holes are placed so that they do not touch, at least seventeen triangles are required to isolate them from a tangram border. Since the seven tans are made of sixteen such triangles, completing the required tangram is impossible. For nonsnug tangrams it appears that no more than three holes not touching one another or the border are possible.

The square is the only snug tangram with all its sides irrational. This is how Read proves it. As I explained in the previous chapter, an irrational tangram drawn on graph paper with the unit square tan oriented orthogonally would have all its edges making 45-degree angles with the matrix lines. All corners clearly must be either 90 degrees or 270 degrees. Since each side is a multiple of $\sqrt{2}$ and the total area is 8, it follows that any irrational snug tangram will be a tetromino composed of four squares, each $\sqrt{2}$ on the side.

There are five tetrominoes. One of them, the square, we know can be formed. Each of the other four is easily proved impossible by placing the square tan in each of three possible positions (*see* Figure 22) and then exploring ways of completing the tetromino. The first tetromino is ruled out at once because there is no way to place the two large triangles. In the other three cases, for each position of the square tan, there are at most only four ways to place the two large triangles. In every case, after the square and two large triangles are placed, there is no spot for the rhomboid. Thus the square is the only possible irrational snug tangram.

My solution of the farm problem, with a limit of 10.985 + , is shown in Figure 23.

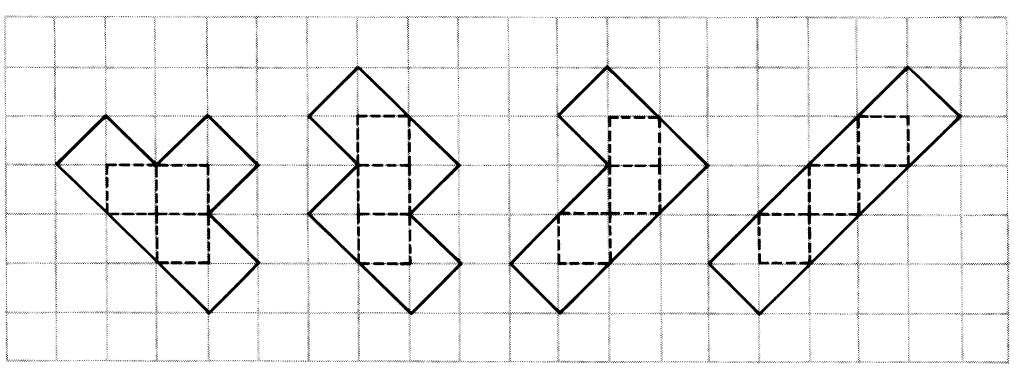

Figure 22 Impossibility proof for the nonsquare tetrominoes

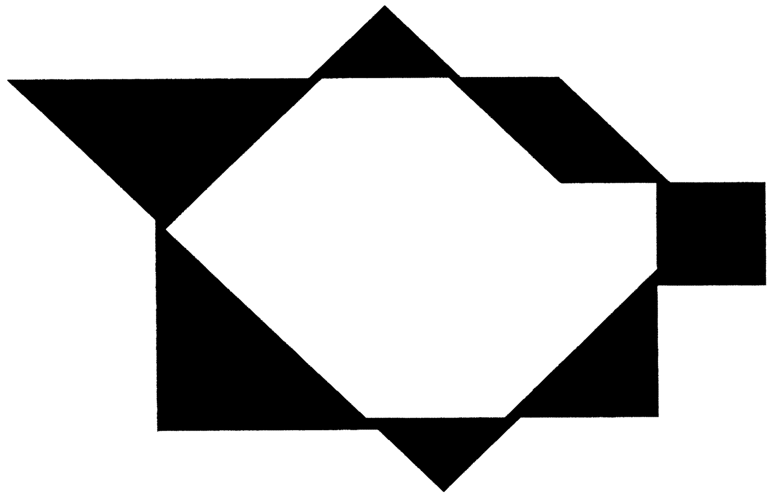

Figure 23 A solution to the farm problem

ADDENDUM

Scores, perhaps hundreds, of puzzles similar to tangrams, but with different dissections of squares, rectangles, circles, and other shapes, have been described in books and articles and manufactured around the world. Some of the marketed variants are pictured in Professor Hoffmann's pioneer book on mechanical puzzles, in *Creative Puzzles of the World* by Pieter Van Delft and Jack Botermans, and in *Puzzles Old and New* by Jerry Slocum and Botermans.

Of special interest, because it predates any known Chinese publication, is a 32-page book published by Kyoto Chobo in Japan in 1742. It gives forty-two patterns to be made with the seven pieces obtained by dissecting a square as shown in Figure 24. The book's title translates as *The Ingenious Pieces of Sei Shonagon.* (Sei Shonagon was a court lady of the late tenth and early eleventh centuries, who wrote the famous *Pillow Book*.) Nothing is known about the author, who uses the pseudonym Ganrei-Ken. It is highly unlikely that Sei Shonagon knew of the puzzle.

Shigeo Takagi, a Tokyo magician, was kind enough to send me a photocopy of this rare book. Unlike the Chinese tans, the Shonagon pieces will form a square in two different ways. Can you find the second pattern? The pieces also will make a square with a central square hole in the same orientation. With the Chinese tans it is not possible to put a square hole anywhere inside a large square.

Richard Reiss, a professor of English at Southeastern Massachusetts University, sent a good proof that no four-sided nonconvex polygon can be made with all

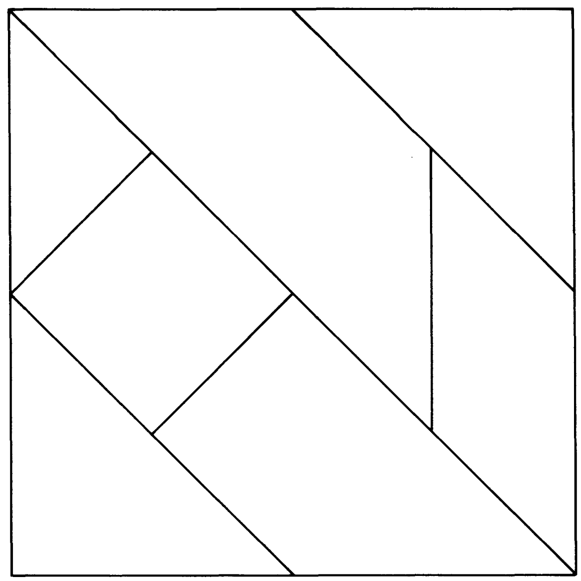

Figure 24 The Sei Shonagon pieces

seven Chinese tans. Peter Van Note proposed the following three tasks based on forming two congruent replicas of a tan:

1. It is possible to make one large square. Use the seven pieces to make two congruent small squares.
2. A large isosceles right triangle is possible. Use the seven pieces to make two congruent isosceles right triangles.
3. A large rhombus is possible. Van Note could not prove it, but he is convinced that the seven pieces cannot make two congruent rhombuses.

John H. Conway wrote from Cambridge University to pose an interesting unsolved problem. What would be the shapes of an "optimal" set of tans—that is, seven convex polygons that will form the largest number of distinct convex polygons?

Figure 25 is taken from Joost Elffers's and Michael Schuyt's marvelous book. The pattern is made with the traditional Chinese tans.

Karl Fulves, author of many books on magic, made the following suggestion for an amusing trick. It involves the tangram paradox at the bottom of

Figure 12. You secretly add to a set of tans a third small triangle. Make the man with the feet, using the pattern shown on the right, and the extra triangle for the feet. Now use sleight of hand to "vanish" one of the small triangles (or just put it in your pocket), then form the man with the foot again, using the pattern on the left. If no one has bothered to count pieces, it seems as if the vanished piece has mysteriously returned. Similar tricks can, of course, be performed with other paradoxical pairs.

Several proposals have been made for using two sets of tangrams to play a board game similar to Solomon W. Golomb's pentomino game. Game inventor Sidney Sackson recommends a 6 × 6 checkerboard, its squares the size of the square tan. Each of two players has a set of seven tans. Players take turns placing any tan on the board, wherever they wish, provided the tan's corners fall on the board's lattice points. The person unable to place a tan loses. Many variations in rules are possible, and larger boards can be used for more than two players.

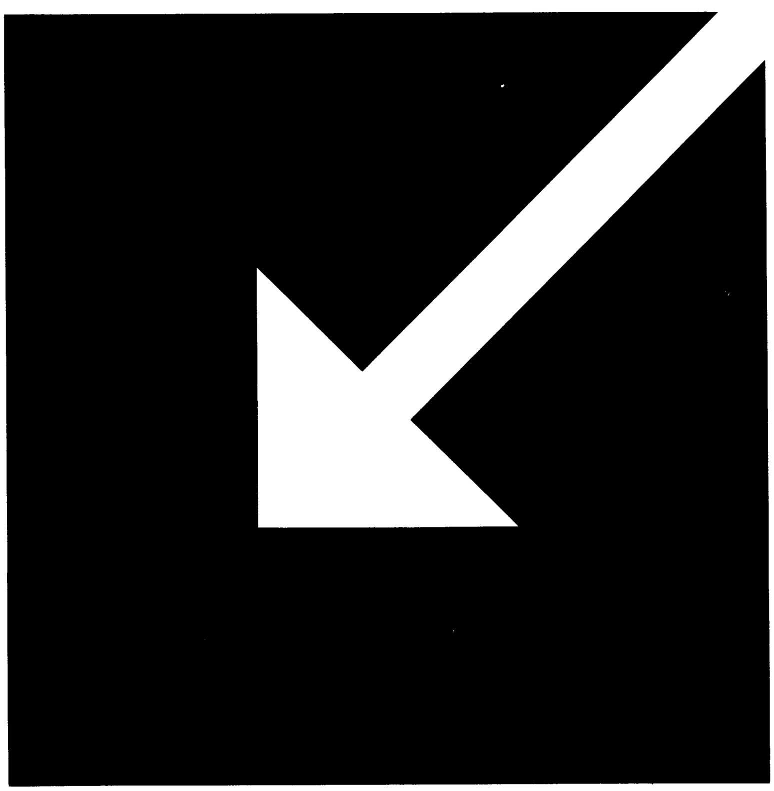

Figure 25 A tangram to solve

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