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ARTICLES

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Have you met M_{13}?

In a short 1997 article [**5**], John Conway describes a sliding-piece puzzle called M_{13}, which bears the same relation to the Mathieu group M_{12} as the famous 15-Puzzle bears to the alternating group A_{15} (the latter summed up neatly by Aaron Archer [**2**]). Much more information on the puzzle appeared in a subsequent, highly enjoyable article by Conway, Elkies, and Martin [**8**]. My first reaction on discovering the M_{13} articles was a somewhat unscholarly exclamation ("Play!"), so I implemented a clickable version of the puzzle on my computer [**16**]. A 15-puzzle veteran, I assumed I was equipped to solve M_{13} with minimal effort, but my endgame was vexed by the tricky moves. Should you find it similarly vexing to solve by hand, you will find a solution in the final section. However, I also found the game an excellent way to learn a bit more about the inner (and outer) workings of two much-renowned groups. The main results of this article will explain which algebraic symmetries of the abstract group M_{12} are compatible, in a certain sense, with the puzzle structure of M_{13} by connecting them to geometric symmetries of the playing surface.

The board for Conway's game is a "projective plane of order 3," which consists of thirteen points and thirteen lines. Each line contains four points, each point belongs to four lines, and any two lines intersect in a unique point. We won't worry about any more general definition of projective planes; you can read all about them in Stevenson's book [**17**] or many others. FIGURE 1 shows the particular 13-point plane that we will be using.

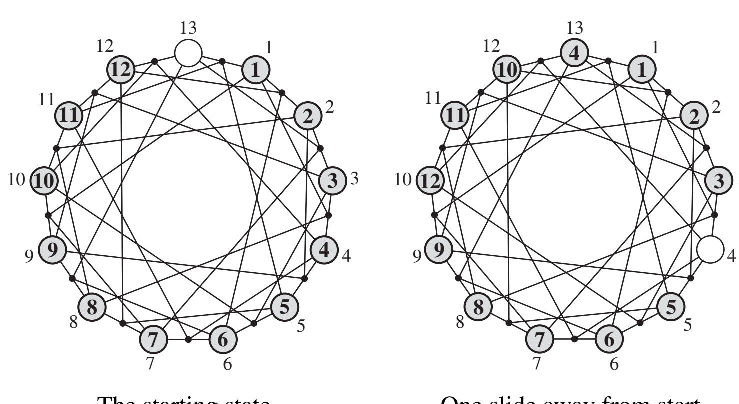

Figure 1 Numbered tiles on a projective plane of order 3

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The points of the plane are numbered 1–13, clockwise around the circle, and points 1–12 begin the game with like-numbered tiles on them. Point 13 begins with no tile upon it, and is thus deemed, for the time being, the "holy point." Lines of the plane are depicted by the dots around the circle, each dot connected to four points of the plane which constitute a line. With our numbering scheme, lines of the plane all have the form $\{i, i+1, i+5, i-2\}$, where i ranges from 1 to 13, and all the numbers are reduced mod 13.

There is only one type of legal move in the game: Choose any point of the plane with a tile on it, and move that tile to the current holy point (which incidentally makes your chosen point the new holy point). At the same time, swap the two tiles on the other points of the line which connects your point to the holy point. FIGURE 1 illustrates what happens if we move tile 4 to the holy point from the initial configuration. Pay attention to points 10 and 12, which complete the line containing 4 and 13.

Conceptually, it can be more convenient to think of the hole moving, and to describe a longer sequence of moves we'll simply list in succession the points the hole visits, in square brackets. For example, FIGURE 2 shows the results of playing two longer sequences, [1, 2, 4, 13] and [1, 2, 13, 6, 12, 13]. Verify these to check your understanding of the rules.

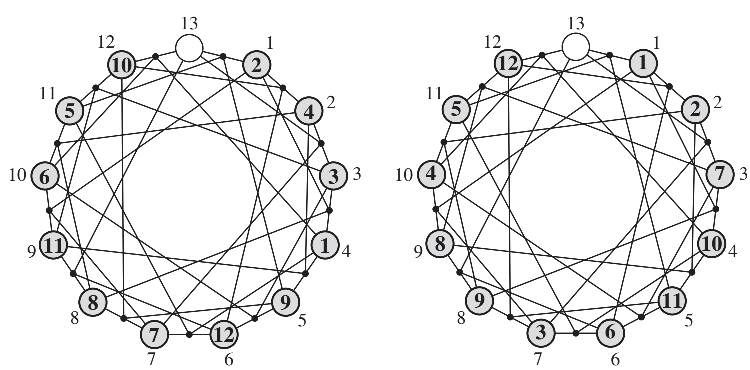


Figure 2 Two move sequences and the resulting permutations of the tiles

Since both of these sequences return the hole to point 13, they can be viewed as permuting the twelve numbered tiles. In cycle notation, the first sequence induces the permutation (1 4 2)(5 11 9)(6 10 12), and the second induces (3 7)(4 10)(5 11)(8 9). Note how to read the cycles in game terms: whatever position you happen to start with, the first sequence will send the tile on point 1 to point 4; the tile on point 4 to point 2; and so on. Sequences such as these which return the hole to point 13 are called *closed* sequences, and the corresponding permutations can be composed to form a subgroup of S_{12}, which we'll call the *game group*, or G for short. If we think of the hole as just another tile numbered 13, to which special rules apply, then we can think of the game group as a subgroup of S_{13} (whose elements all leave the 13th tile fixed). It is properly contained in the set of all permutations in S_{13} that can be achieved by legal moves. This larger set is what Conway refers to as M_{13}. M_{13} is not a closed set under composition of permutations (the rules of the game place restrictions on which permutations can be legitimately composed), and it is not our primary interest here, although it was this larger set which originally motivated the game.

The point of all this as a *puzzle*, of course, is to start from a scrambled position such as the ones in FIGURE 2 and find a sequence of moves that returns the tiles to their initial positions. The slippery, indirect moves make this a challenge to solve intuitively, even though the puzzle has far fewer positions than other familiar permutation puzzles such as Rubik's Cube or the Fifteen Puzzle mentioned above.

One game, two simple groups

M_{13} and the 15-puzzle both generate interesting permutation groups as their pieces are scrambled. But compared to the rectangular grid of the 15-puzzle, the plane on which M_{13} tiles move has a rich group of symmetries in its own right. It is primarily the interplay between that group of geometric symmetries and the game group G (determined by the tile-sliding rules) which I want to explore. The next two theorems gather up the basic facts about these two groups.

THEOREM 1. *The game group is isomorphic to the Mathieu group M_{12}, a simple group of order* 95,040 *which acts sharply* 5*-transitively on a* 12*-element set.*

Just what is M_{12}? Part of the reason it still fascinates us more than a century after its discovery is that there are many equally good answers to that question, involving everything from twisting and sliding puzzles to card shuffling to codes. Joyner gives a nice view of M_{12}'s many faces [**12**], and there is still more in Conway and Sloane's weighty reference [**7**]. For the present purposes, it suffices to know that M_{12} is our game group, and that its deep and varied connections make it worth exploring from any accessible angle.

The isomorphism assertion connecting this game to other definitions of M_{12} is proven in the paper of Conway, Elkies, and Martin [**8**]. "Acts sharply 5-transitively" simply means the following: choose your favorite five tiles and your favorite five points for them to live on (an ordered choice). Then there is one and only one position in the game that puts your chosen five tiles on your chosen five points. For example, FIGURE 2 shows the one and only legal position with tiles 2, 4, 3, 1, and 9 on the first five points (in that order). This may sound like an unremarkable property for a group to have, but quite the opposite is true: The property of acting sharply 5-transitively on a 12-element set characterizes this group uniquely up to isomorphism. Dixon and Mortimer [**9**] give a nice summary of sharply k-transitive group actions, in addition to their easy-to-follow construction of M_{12} and the other Mathieu groups.

We can (and we will) understand some features of the game group better if we also pay attention to the symmetries of the playing surface—the projective plane. That gives us another set to consider inside S_{13}, namely, the permutations in S_{13} which preserve the plane's structure (entirely independent of whether they can be achieved by legal moves in the game). Preserving structure in this sense means that lines must be mapped to lines, and permutations that do this will be called *plane automorphisms*. For example, the picture of the plane we've used in FIGURE 1 makes at least one small family of plane automorphisms clear: we can simply rotate the figure one or more clicks clockwise, inducing a 13-cycle of the points which clearly sends each line to another line. Such permutations can be composed to give a group, which we'll call the *plane group*. It is known by various notations, including $PGL(3, 3)$, which is most suggestive of its role here. Much the way that an ordinary linear transformation can be described by its action on a basis, an element of our plane group can be described by its action on a suitable set of points in the plane: an *ordered oval* is an ordered set of 4 points in the plane, no 3 of which belong to a line.

THEOREM 2. *The plane group is a simple group of order* 5,616 *which acts sharply transitively on ordered ovals.*

This group belongs to an infinite family of finite simple groups, the projective special linear groups, and you can find an accessible proof that they are simple in the textbook of Beachy and Blair [**3**]. The transitivity assertion is proven in an elementary way in [**8**].

In both of the preceding theorems, the order of the group can be inferred from the transitivity assertion, as follows. In the case of M_{12}, any given selection of 5 tiles can be put in $12 \times 11 \times 10 \times 9 \times 8 = 95{,}040$ different positions (and elements of the group are in one-to-one correspondence with such ordered selections). In the case of the plane group, you need to count the number of ordered ovals: choose any two points (13×12), choose a third point off the line they determine (9 possibilities), and one more point off the three lines that have been determined so far (4 possibilities); all together, $13 \times 12 \times 9 \times 4 = 5{,}616$ ordered ovals, in one-to-one correspondence with elements of the plane group. The plane group also acts transitively on the points of the plane (this follows from transitivity on ovals, or more simply from the order 13 rotational symmetry we've already pointed out), so the stabilizer of any particular point—say, number 13—is a subgroup of order $5616/13 = 432 = 2^4 \cdot 3^3$.

Geometrically related sequences

As an example of interaction between the game and the plane group, let's look at some ways we can transform move sequences while preserving the cycle structure of their permutations.

An easy first example of such a transformation is the reversal of sequences: The inverse of $[a_1, a_2, \ldots, a_n, 13]$ is achieved by $[a_n, a_{n-1}, \ldots, a_1, 13]$, so these two sequences induce permutations of the same cycle type. The reversal would look more literal if I followed the convention of Conway and included a 13 at the beginning of the sequence to indicate the starting position of the hole, but I will still just refer to this as reversal.

Earlier I suggested that we might think of the hole as simply an invisible "tile number 13," to which special rules apply. We can easily imagine a variant game in which another numbered tile is the special one—call it an x-type game if tile x plays the "holy role." With this in mind, consider a move sequence $S = [a_1, a_2, \ldots, a_n]$ and a plane automorphism ϕ. We can apply ϕ term-by-term to S to obtain $\phi(S) = [\phi(a_1), \phi(a_2), \ldots, \phi(a_n)]$. If S is a closed sequence, inducing an element in the game group, then $\phi(S)$ is a closed sequence for a $\phi(13)$-type game, and would induce an element in the corresponding game group. This would have the same cycle type as S—the cycles of $\phi(S)$ are merely the cycles of S, "relabelled" by ϕ. We could say that two sequences related by a plane automorphism in this way are isomorphic sequences; this is an equivalence relation, and what we have just said as that isomorphic sequences induce the same type of permutation.

There is a weaker equivalence relation, slightly less obvious, which still gives the same implication for induced permutations. Consider a move sequence for the basic game, say,

$$S = [a_1, a_2, \ldots, a_n, 13]$$

If we rotate this sequence one step to the left to obtain

$$S_1 = [a_2, a_3, \ldots, a_n, 13, a_1],$$

we see that S_1 is a closed sequence for an a_1-type game. It will induce a permutation in that game of the same type as S. To see this, it helps to temporarily adopt an expanded notation for moves—let a parenthesized pair (xy) denote sliding the hole from x to y. Using this notation,

$$\begin{aligned} S_1 &= (13a_1)(a_1a_2)(a_2a_3)\cdots(a_n13) \\ &= (a_113)(13a_1)(a_1a_2)(a_2a_3)\cdots(a_n13)(13a_1) \\ &= (a_113)\ S\ (13a_1), \end{aligned}$$

and so S_1 is simply a conjugate of S, which will induce a permutation of the same cycle-type (on an a_n-type game) as S does on the ordinary game. This can be iterated; if S_i denotes the move sequence S rotated i terms to the left, then S_i is a closed sequence for an a_i-type game which induces a permutation of the same cycle type as S does on the ordinary game.

PROPOSITION 1. *If S and T are closed sequences (for the standard game) and T can be obtained from S by (1) applying any plane automorphism to the terms of S, (2) reversing and/or rotating the resulting move sequence any number of positions left or right, then T and S induce permutations of the same cycle type.*

This gives each move sequence for the ordinary game a large class of sequences of the same length which behave “the same,” at least as far as cycle structure. This can be helpful in reducing the amount of computation needed to study move sequences of moderate length.

Plane automorphisms within the game group

Once I learned to play the game well and unscramble the tiles reliably, I began looking for other amusements, and the question of realizing plane automorphisms via legal moves in the game was a challenge that occurred naturally—somewhat like the pastime of constructing pretty patterns on Rubik’s cube. The prettiest patterns on the cube, generally, are the ones that capitalize on its symmetry in some way.

Of course, most permutations in M_{13}’s game group are not plane automorphisms. The first example in FIGURE 2 is typical: the line connecting points 1 and 2 in the plane now contains tiles $\{2, 4, 10, 12\}$ which do *not* constitute a line of the plane. Of course, that must be true simply by cardinality (the game group is far larger than even the entire plane group), but experimenting with short move sequences might make you wonder whether there are *any* legal moves which send the tiles of each line to the points of another line. Consider the second example in FIGURE 2, however. If you find it tedious to inspect all thirteen lines to verify that this move does indeed result in a plane automorphism, a suitable drawing of the plane (FIGURE 3) can reveal it “at a glance” as a reflection of the plane across the line $\{1, 2, 6, 12\}$. In fact, any sequence of the form $[a, b, 13, c, d, 13]$ where $\{a, b, c, d\}$ form a line *not* incident with point 13 will induce a plane automorphism with a similar reflection picture. In FIGURE 3, we’ve taken the liberty of doubling points on the line $\{2, 3, 7, 13\}$ in order to put as much symmetry on display as possible. If you like, you can imagine stitching the boundary together so that antipodal points are identified, eliminating the doubling—and incidentally embedding our diagram in the real projective plane $\mathbb{R}P^2$—but you’ll have to imagine it on very stretchy material. The model of the plane used in FIGURE 3 can be found in many references on projective planes. Polster’s book [**14**] gives it and several other useful pictures that we’ll be using here.

Once you’ve found a few automorphisms in the game group, it’s hard to resist wondering just how many you can find. I’d like to answer that question gradually, starting with a very distinguished set. The paper of Conway, Elkies, and Martin [**8**] identifies

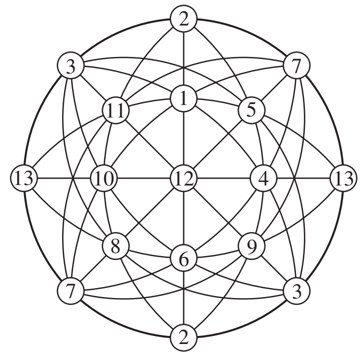


Figure 3 Projective plane exhibiting reflective symmetries

eight positions in the game group which require the maximum number of moves—nine moves, as they show there—to achieve from the start position (or equivalently, to return to the start position). Together with the identity, those eight form an abelian group of order 9, in which every nonidentity element has order 3. We denote this group T; it is generated by

$$\theta = [9, 1, 13, 2, 4, 3, 7, 10, 13] = (1\ 5\ 11)(6\ 9\ 8)(4\ 10\ 12)$$

and

$$\omega = [10, 1, 11, 12, 5, 13, 7, 9, 13] = (2\ 3\ 7)(4\ 10\ 12)(6\ 8\ 9).$$

Once it occurs to you to check, you can verify that θ and ω are both elements of the plane group, but in the case of θ, I hope the picture of the plane in FIGURE 4 is more enlightening than a line-by-line audit. Several points (numbers 2, 3, and 7) may appear to be missing from FIGURE 4, but they are tucked away behind 13 on an axis perpendicular to the printed page. Also, the circular "lines" each seem to be missing a point, but each of them contains one of the points 2, 3, or 7 on that central axis. This picture is uncomfortable in two dimensions; it would prefer to be seen as a torus, with the fixed points on the central axis and other lines of the plane as "lines" of slope 0, 1, and -1 wrapping around it. But either way you picture it, there is a rotational symmetry of order 3 which is precisely θ, and you can provide a similar picture to show that ω can also be considered as an order 3 rotational symmetry of the plane. It follows that all the elements of the "maximum depth group" T belong to the plane

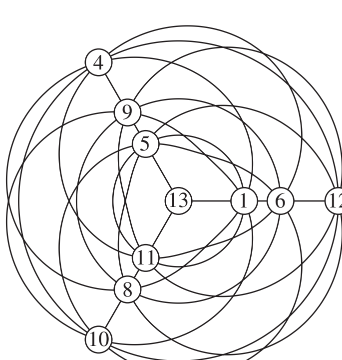


Figure 4 Rotational symmetry of order 3 about a fixed line

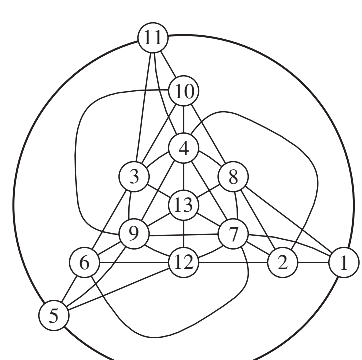


Figure 5 Rotational symmetry of order 3 about a single fixed point

group as well, and they all have similar geometry: one line through point 13 is fixed (pointwise) as the axis of rotation, while the other three points in each of the remaining lines through point 13 move in a 3-cycle. Now, consider one more move sequence and its associated permutation,

$$\zeta = [4, 9, 11, 6, 8, 10, 5, 13] = (1\ 11\ 5)(2\ 10\ 6)(3\ 12\ 8)(4\ 9\ 7),$$

and yet another picture of our plane (FIGURE 5) to show that ζ is also a plane automorphism.

This ζ normalizes the group T generated by θ and ω—you can check this directly, or see Proposition 3 below. All together, then, they generate a group of order 27 which belongs to the intersection of the game group and the plane group. Since $432 = 2^4 \cdot 3^3$, we have found a Sylow 3-subgroup in that intersection.

In constructing a Sylow 2-subgroup in the intersection, we will again rely on figures. FIGURE 6 shows one more, partially completed model of the projective plane. Rotating the hook-shaped "line" through multiples of $\pi/4$ supplies the remaining lines of the projective plane, which I'll omit for clarity.

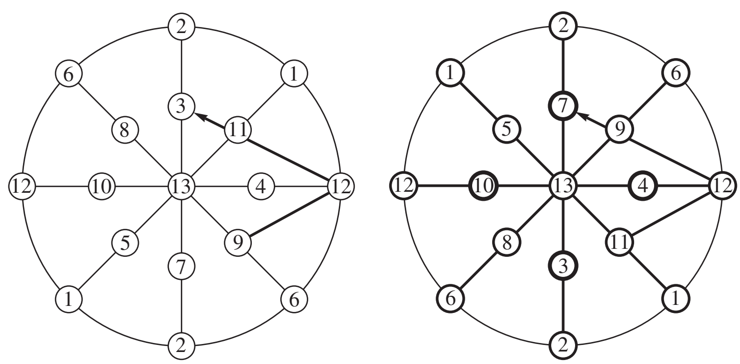


Figure 6 Partial diagram of the plane. Reflection doesn't carry lines to lines...

The completed model obviously has a rotational symmetry of order 8; call it ρ. In cycle form, $\rho = (1\ 2\ 6\ 12)(3\ 8\ 10\ 5\ 7\ 9\ 4\ 11)$, and this can be achieved in the game group as $\rho = [7, 6, 8, 12, 6, 3, 4, 13]$. The figure may appear to have reflective symmetry across the four lines that appear as diameters of the circle, but this is not the case. Simply reflecting points across, say, the horizontal diameter as shown, we find that the line $\{9, 12, 11, 13\}$ has *not* been carried to another line of the plane. The lines in the model have a counterclockwise orientation, if you follow them from short end to long end as indicated by the arrowheads. Reflection reverses that orientation. However, we can fix this with just a small adjustment: give the four emphasized points in FIGURE 6 an additional 180° twist, and all the lines have been restored to their proper orientation in FIGURE 7. I have left just one of the hook-shaped lines in the diagram; you can pencil in its rotations to check that they are actually lines of the plane.

Let's call this reflection-with-a-twist $\tau = (1\ 6)(5\ 8)(9\ 11)(4\ 10)$. It belongs to the game group as, for example, $\tau = [12, 7, 3, 12, 2, 13]$. You can compute that $\tau\rho\tau^{-1} = \rho^3$ (use the picture!), so τ and ρ generate a group of order 16 which is a Sylow subgroup in the intersection of the game group and the plane group. If τ were literally a reflective symmetry of this model, we would have the familiar dihedral group of order 16; as it is, our reflection-with-a-twist generates what is known as the *quasidihedral group* or *semidihedral group* of order 16. Usage varies, but the latter term is used by

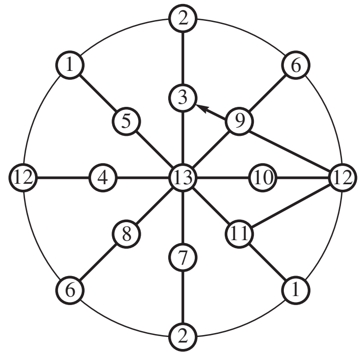


Figure 7 ... but that can be corrected with an additional twist.

Wild [**18**] in his round-up of groups of order 16. The projective plane diagrams above are the simplest geometric representation I know for this group. And with plane automorphism groups of order 3^3 and 2^4 inside the game group, we have proven the following:

PROPOSITION 2. *The game group contains all* 432 *plane automorphisms which leave point* 13 *fixed.*

For the remainder of the paper, let P denote this 432-element subgroup of the game group. P can be presented with just two generators; ζ and τ will suffice, in fact, and with access to computer algebra systems it is easy to find such pairs of generators—but it is also satisfying to see automorphisms which generate the Sylow subgroups, concretely and geometrically. Both the game group of order 95040 and the plane group of order 5616 can also be presented with two generators. See the short articles of Leech [**13**] and Bisshopp [**4**] for specific generators, if you like.

A few terms and definitions

These plane automorphisms within the game group play a special role in the following sections, where we will study group automorphisms of M_{12} and how they relate to the extra structure provided by the rules of Conway's game. A quick review of some standard definitions and facts may be useful before we go on. Since we've committed the letter G to denoting our game group, let H stand for any finite group.

The set $Z(H) = \{x \in H | xh = hx, \forall h \in H\}$ is known as the *center* of H. It's routine to check that $Z(H)$ is a normal subgroup of H, so, in particular, the center of a nonabelian simple group like G is trivial.

The automorphisms of H themselves form a group under composition, which is denoted $\mathrm{Aut}(H)$. An *inner automorphism* of H is an automorphism of the form $x \mapsto hxh^{-1}$ ("conjugation by h") for some fixed $h \in H$. These form a subgroup of $\mathrm{Aut}(H)$, denoted $\mathrm{Inn}(H)$. There is a surjective homomorphism from H to $\mathrm{Inn}(H)$ (namely, mapping each $h \in H$ to the "conjugation by h" automorphism), the kernel of which is none other than $Z(H)$. Thus, if $Z(H)$ happens to be trivial (as in the case of G), then this mapping is injective; each element of H represents a distinct element of $\mathrm{Inn}(H)$.

Elements of $\mathrm{Aut}(H)$ which are not inner automorphisms are called *outer automorphisms*. A final routine fact is that $\mathrm{Inn}(H)$ is a normal subgroup of $\mathrm{Aut}(H)$, and the quotient group $\mathrm{Aut}(H)/\mathrm{Inn}(H)$ is denoted $\mathrm{Out}(H)$, the *outer automorphism group* of H. Confusingly, the elements of $\mathrm{Out}(H)$ are not outer automorphisms; they are equivalence classes of automorphisms. Two elements α and β in $\mathrm{Aut}(H)$ represent the same class in Out(H) if and only if α can be expressed as the composition of β with some inner automorphism of H.

With definitions in hand, it is now time to identify, precisely, the "extra structure" that I have been referring to.

Depth and automorphisms of the game group

The *depth* of a position in the game group is the minimum number of moves required to solve that position (or, equivalently, to achieve it from the initial state). Depth is not a purely group-theoretic notion. It comes, essentially, from the choice of generators that we are allowed by the legal moves of the game—and that means that there's no reason to expect automorphisms of the game group to preserve depth. Elements of the subgroup P, however, do give us examples:

PROPOSITION 3. *Conjugation by a plane automorphism in the game group preserves depth: That is, if $\phi \in P$ and $\alpha \in G$, then α and $\phi\alpha\phi^{-1}$ have the same depth.*

Proof. Let $S = [a_1, a_2, \ldots, a_n, 13]$ be a sequence of moves that induces α, and consider $\phi(S) = [\phi(a_1), \phi(a_2), \ldots, \phi(a_n), 13]$. It's easy to see how $\phi(S)$ must act on the tiles: if S carries the tile on point x to point y, then $\phi(S)$ carries the tile on point $\phi(x)$ to point $\phi(y)$. But that is precisely how $\phi\alpha\phi^{-1}$ acts, so $\phi(S)$ is a sequence that achieves $\phi\alpha\phi^{-1}$. Thus for every sequence that achieves α there is a sequence of the same length that achieves $\phi\alpha\phi^{-1}$, and vice versa of course, using ϕ^{-1} in place of ϕ. ■

In the present case, conjugation by elements of P provides 432 inner automorphisms of G which preserve depth (432 *distinct* automorphisms, in fact; you can deduce this from the fact that G is simple). And that's all there are:

PROPOSITION 4. *G has precisely* 432 *inner automorphisms which preserve depth.*

Proof. Since the previous proposition shows there at least 432, we only have to show that there are no more than 432. Consider the element

$$\theta = [9, 1, 13, 2, 4, 3, 7, 10, 13] = (1\ 5\ 11)(6\ 9\ 8)(4\ 10\ 12)$$

We have already mentioned that all the (nonindentity) elements of the maximum-depth group T have the same cycle structure as θ—the fixed points constitute a line through point 13, and each of the remaining lines through 13 has its three remaining points permuted in a three-cycle. Thus, if conjugation by α carries θ to another element of T, α must carry the set $\{1, 5, 11\}$ to a set of three points which are collinear with 13, and there are 12×2 ways to do this. Once the image of that set is determined, there are only six choices for $\alpha(6)$ and then three choices for $\alpha(4)$. And at this point, if α belongs to G, it is uniquely determined, since G is sharply 5-transitive. Thus there are at most $12 \times 2 \times 6 \times 3 = 432$ such elements α in G. ■

Outer automorphisms

Our game group has more than inner automorphisms, but not much more; it is a fact that the outer automorphism group of M_{12} has order two [**6**]. That means any two outer automorphisms of M_{12} represent the same class in Out(M_{12}), and differ from one another only by composition with some inner automorphism. Conway et al. [**8**] show how one such outer automorphism can be discovered using a "dualized" version of the M_{13} game in which one slides lines as well as points. We will not duplicate that exposition here, but simply rely on their work to produce an explicit representative of the nontrivial class of outer automorphisms. To describe our automorphism, we'll need a set of generators for the game group. The following two (chosen fairly arbitrarily) suffice.

$$x = [4, 11, 3, 4, 13] = (1\ 8)(3\ 11)(6\ 7)(9\ 10)$$

and

$$y = [8, 7, 11, 3, 4, 12, 1, 13] = (1\ 8\ 10)(2\ 6\ 4)(3\ 5\ 9)(7\ 12\ 11)$$

We won't verify directly that they are generators, but they satisfy the relations for the presentation given in the online *Atlas of Finite Group Representations* [**1**]. By following the dualization construction (making the necessary adjustments for the different labeling of points in our projective plane) we can produce a specific outer automorphism of the game group, namely F, which acts on the generators above by

$$F(x) = (1\ 10)(2\ 8)(3\ 6)(5\ 11)$$

and

$$F(y) = (1\ 12\ 6)(4\ 9\ 5)(8\ 11\ 10)$$

You can check that these satisfy the same relations as x and y, so we do have an automorphism (it's easily checked once it's presented, but finding it would be difficult without a clever construction such as dualizing). Is it depth-preserving? Recall our maximum-depth element $\theta = (1\ 5\ 11)(6\ 9\ 8)(4\ 10\ 12)$ from the section on plane automorphisms. We can express θ in terms of our generators, $\theta = (xy)^3 y(xy)^{-3} y^{-1}$, from which we can compute $F(\theta) = (1\ 12\ 6)(4\ 9\ 5)(8\ 11\ 10)$, which is not one of the eight elements in subgroup T, so F is not depth-preserving. But note that a similar calculation with generators shows that $F(\omega) = \theta$, so F does send ω to another element at depth 9.

It is still plausible that some other outer automorphism of G might be depth-preserving. A little computer assistance helps to rule this out; the computer can easily give us a list of the conjugacy classes and their sizes in the game group. The list of 15 classes in TABLE 1 shows each class's cycle type in addition to its size. The same information is available in the *Atlas* [**6**].

TABLE 1: Conjugacy classes in M_{12} (arbitrarily numbered)

Class	Cycle type	Size	Class	Cycle type	Size
1	Identity	1	9	3^4	2640
2	2^4	495	10	4^2	2970
3	2^6	396	11	$4 \cdot 8$	11880
4	$2^2 \cdot 4^2$	2970	12	5^2	9504
5	$2 \cdot 3 \cdot 6$	15840	13	6^2	7920
6	$2 \cdot 8$	11880	14	11	8640
7	$2 \cdot 10$	9504	15	11	8640
8	3^3	1760			

Now, consider an element from the 4^2 class, such as $z = [2, 10, 3, 7, 8, 13] = (5\ 9\ 6\ 12)(7\ 10\ 8\ 11)$. By expressing z in terms of x and y (a little tedious) we can compute that $F(z) = (1\ 12)(7\ 10)(2\ 3\ 8\ 11)(4\ 5\ 9\ 6)$, in the $2^2 \cdot 4^2$ class, so F exchanges these two classes, and every outer automorphism of G must do the same. Those classes, as it turns out, both contain elements at depth 6 and 7 only. But they differ in their distribution. See TABLE 2, which was computed with Mathematica. Thus, every outer automorphism of G moves elements from depth 6 to depth 7 and vice versa. And with that, we have proven:

TABLE 2: Distribution by depth for two conjugacy classes

Class	Cycle type	Depth 6	Depth 7
4	$2^2 \cdot 4^2$	972	1998
10	4^2	1188	1782

PROPOSITION 5. *The game group G has precisely* 432 *automorphisms which preserve the depth of every element in the group, namely, conjugation by the* 432 *elements of the plane automorphism subgroup P.*

We can now deduce one further result which identifies the subgroup T as the essential "test case" for depth-preserving automorphisms. The proof of Proposition 4 actually shows that an inner automorphism which preserves the subgroup T preserves the depth of every element in G. And as we have remarked, any outer automorphism of M_{12} is simply the composition of our particular F with some inner automorphism. Consider F followed by conjugation with some element g. $F(\omega) = \theta \in T$, so if $gF(\omega)g^{-1}$ also belongs to T, then we know that g is a plane automorphism, and $gF(x)g^{-1}$ has the same depth as $F(x)$ for every $x \in G$. In particular, we computed that $F(\theta) \notin T$, so $gF(\theta)g^{-1} \notin T$. Thus, every outer automorphism sends some element of T, either ω or θ, to an element of strictly lower depth, and we have proven the following tidy (and surprising) result.

PROPOSITION 6. *An automorphism of the game group G preserves the depth of all elements if and only if it preserves depth for the eight maximal-depth elements in the group.*

It would be interesting to see if a similar result holds for other permutation puzzles, such as the Skewb or the Pyraminx, which have a small number of positions at the maximum depth. These puzzles are described in Hofstadter's popular article [**11**] and Joyner's more recent book [**12**], but the best source for information about the distribution by depth in these puzzles is online [**15**].

Solving M_{13} by hand

In one sense, the "optimal" algorithm for solving the puzzle is what puzzlers sometimes refer to as God's Algorithm (this term was at least popularized, if not originated, by Hofstadter [**10**])—since there are a finite number of positions, simply memorize the shortest solution for each one! This is not practical for human puzzlers, however, and the algorithm given by Conway et al. [**8**] is also not practical if we don't wish to rely on computerized assistance. What follows is an admittedly idiosyncratic set of moves that can be memorized easily and will suffice to unscramble the puzzle from any scrambled state (provided that the scrambled state was obtained by legal moves in the first place). In order to guarantee visible progress as we proceed through the puzzle we will position tiles one at a time, and rely on the magic of sharp transitivity to ensure that the puzzle is solved precisely when we slide the fifth tile into its proper position. Of course, at each step we want to preserve the tiles we have already positioned correctly, and that means we work in progressively smaller subgroups—the stabilizers of our growing set of correctly positioned tiles.

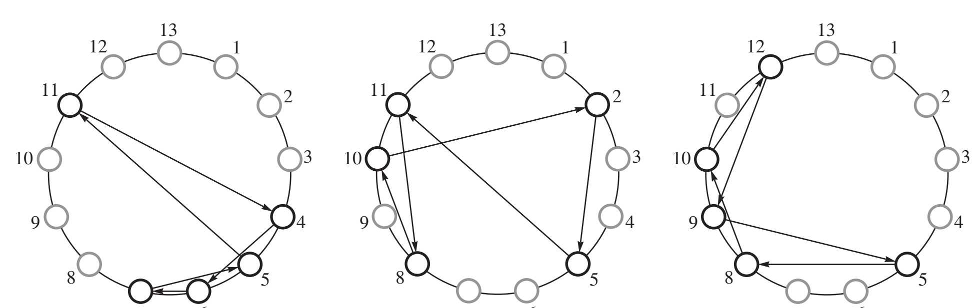
Stage 1: Position tile 1 That is, position tile 1 on point 1, leaving point 13 unoccupied. This can be done intuitively from any position with a closed sequence of just 3 moves, and is left as an exercise to ensure familiarity with the rules of the game. Although this is simple, you'll find you need to vary your strategy depending on whether the point which holds tile 1 is colinear with points 1 and 13, or not (with the colinear case requiring a bit more indirection).

Stage 2: Position tile 3 From this point on, we will only use move sequences that fix the tile on point 1. That restricts us to a subgroup of order $11!/7!$—which is, incidentally, the group known as M_{11}. Since 11 is a prime divisor of the order, there must be an element of order 11 in the group by the well known theorem of Cauchy. Such an element could only act as an 11-cycle on tiles 2–12. Thus the puzzler who has a rabid preference for minimizing the number of cases to consider need only find a short sequence of moves which induces such an 11-cycle, and repeat that sequence until tile 3 is returned home. For example, either of the sequences

$$[4, 5, 6, 7, 8, 13] \quad \text{or} \quad [2, 4, 6, 8, 10, 13]$$

is very easy to remember and induces an 11-cycle leaving the tile on point 1 fixed. Simply repeat until tile 3 has returned home. Realistically, though, this stage requires only 3 or 4 moves, and can be carried out (after a little practice) without relying on memorized sequences.

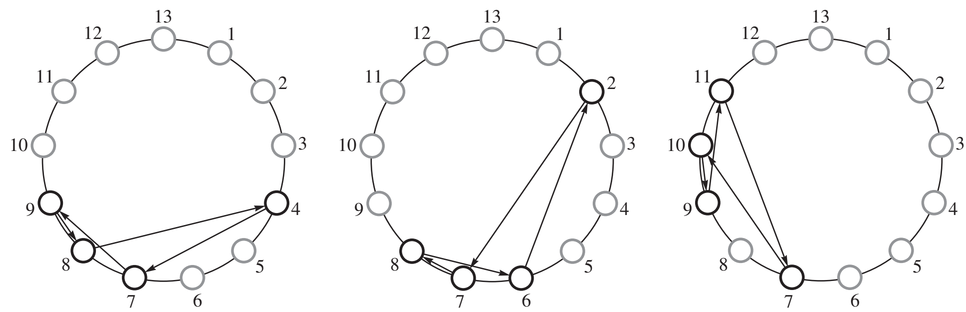
Stage 3: Position tile 5 One of the sequences in FIGURE 8 can be applied (perhaps repeatedly) to return tile 5 home while preserving the tiles on points 1 and 3. Each of them actually induces a *double* 5-cycle, but the illustration shows only the cycle which moves tiles to and from point 5. Simply choose a cycle which contains tile 5 and repeat the sequence until it returns home.



$\alpha = [4, 5, 7, 6, 13]$ $\beta = [8, 5, 2, 10, 13]$ $\gamma = [4, 2, 11, 6, 13]$

Figure 8 Move sequences for Stage 3.

Stage 4: Position tile 7 Very similar to the previous stage: the three sequences in FIGURE 9 almost suffice to return tile 7 home. Again, each of them induces a *double* 4-cycle, but we've only shown the relevant part in the figure. The sequences are quite easy to remember, all having the form $[1, x, 5, 11, y, 13]$.



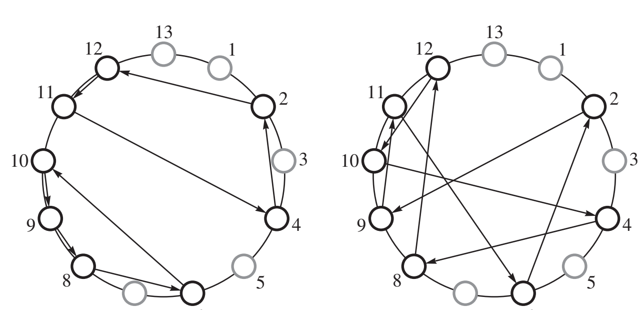
$\rho = [1, 12, 5, 11, 6, 13]$ $\sigma = [1, 12, 5, 11, 4, 13]$ $\tau = [1, 7, 5, 11, 10, 13]$

Figure 9 Move sequences for Stage 4.

Three 4-cycles don't quite suffice to cover all the possibilities. If tile 7 happens to be on point 12 at this stage, you can execute [8, 10, 13] to move it safely to point 4, and then apply ρ. (Since the stabilizer of $\{1, 3, 5\}$ has order 72, we can't use 5-cycles at this stage. There are elements of order 6 in the stabilizer, and you might hope to find a pair of 6-cycles which would cover all the possibilities at this stage, but a quick search on the computer shows this is not possible.)

Stage 5: Position one more tile At this stage we are trying to preserve the position of tiles 1, 3, 5, and 7, which means we are working in a subgroup of only 8 elements! The two sequences in FIGURE 10 suffice to generate the group (and it's easy to check that the group they generate is isomorphic to the familiar quaternion group of 8 elements). If some power of i or j will suffice to return any tile home, great! Otherwise execute i, and then either j or j^{-1} will complete the solution. If you should find that you have five tiles on the correct points but the puzzle still isn't solved, then it never will be solved by further legal moves; it's in a state that lies outside the game group. As a little optimization, note that i^2 and j^2 induce the same permutation of order 2, which can be achieved more efficiently by the sequence [8, 10, 13].

Working down through stabilizers in this way is a simple-minded strategy and is generally very far from providing the shortest possible solution. However, I do hope



$i = [6, 3, 9, 5, 10, 13]$ $j = [4, 1, 12, 3, 8, 13]$

Figure 10 Move sequences for Stage 5 you will have the opportunity to try it with a playable model of the puzzle. I have referred to the "magic" of sharp 5-transitivity, and the feeling that tiles are conspiring to come into place just as I make my last move is still eerie, even though I have seen, on paper, the proof that they must do so. M is for *Mathieu*, naturally, but a little time playing with the M_{13} puzzle can remind us to keep words like *marvelous* and *magical* in mind as well.

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Summary We analyze a sliding tile puzzle (due to J. H. Conway) which gives a presentation of the Mathieu group M_{12}, one of the sporadic simple groups. We use the symmetries of a finite projective plane to classify automorphisms of M_{12} which preserve the depth of puzzle positions, and give an algorithm for solving the puzzle by hand.

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