Math Horizons

ISSN: 1072-4117 (Print) 1947-6213 (Online) Journal homepage: https://www.tandfonline.com/loi/umho20

A Tricoloring Puzzle

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To tricolor a pyramid of hexagonal cells, assign each cell one of the colors red, yellow, or green according to a single rule: each cell taken together with the two cells *directly below it* must be either (1) all three of the same color or (2) three different colors. The coloring on the left in figure 1 is correct while the one on the right is not quite valid, as demonstrated by the marked cells.

Figure 1. Valid and invalid colorings of a five-row pyramid.

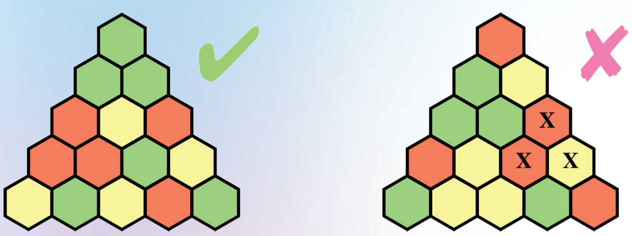


Figure 2. Four warm-up puzzles.

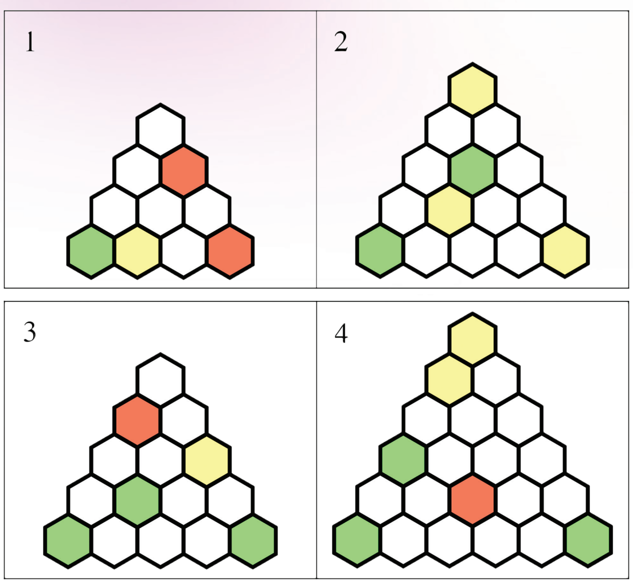
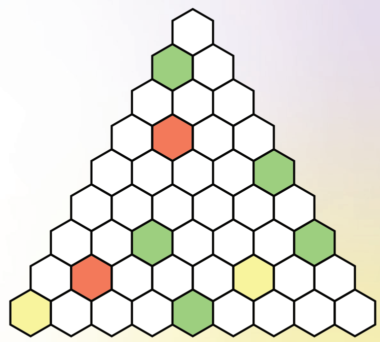
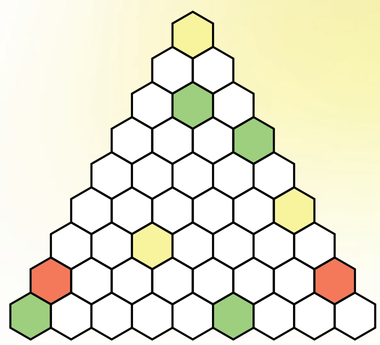


Figure 2 has four puzzles to try as a warm-up. Each one has a unique red/yellow/green coloring that satisfies the single rule. Once you have completed the warm-up puzzles, try your hand at one of the two nine-row puzzles.

If you find these examples interesting and want more to solve, you'll find others at *http://homepages.gac.edu/~jsiehler/games/pyramids-start.html*.





The solution to the nine-row puzzles can be found on page 26.

***Jacob Siehler** is an assistant professor of math at Gustavus Adolphus College. Read more about him and the mathematics involved in creating and/or solving these puzzles in his article "Tricolor Pyramids" on page 23.*

10.1080/10724117.2019.1676106

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To cite this article: Jacob Siehler (2020) Tricolor Pyramids, Math Horizons, 27:3, 23-26, DOI: 10.1080/10724117.2019.1676117

To link to this article: https://doi.org/10.1080/10724117.2019.1676117

Published online: 13 Jan 2020.

Tricolor Pyramids

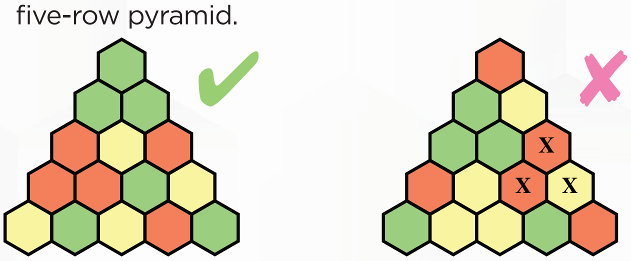
Jacob Siehler

Have you tried the coloring puzzles inside the front cover yet? These puzzles ask you to fill in a pyramid of hexagonal cells with three colors in such a way that each subpyramid of three cells (one on top and two on the bottom) has the property that all three cells are the same color or all three cells are different colors. Figure 1 shows both a valid and invalid coloring.

As with similar logic puzzles, the puzzle designer must place clues carefully to create a puzzle that is not too hard and especially not too easy. The front-cover puzzles, and those online (*http://homepages.gac.edu/~jsiehler/games/pyramids-start.html*), have been chosen with care. While computer assistance is helpful in creating puzzles, understanding the underlying mathematics before jumping into computation can make the puzzles more interesting.

Consider some basic questions that a would-be puzzle designer should be able to answer: what is the minimum number of clues required to uniquely determine the solution to a puzzle, and when does a given set of clues suffice to determine a solution at all? Puzzle solvers will also want to know if there are methods to find solutions without guesswork and backtracking. At first glance, it might not even be clear what branch of math would help to answer these questions: logic? graph theory? combinatorics?

Figure 1. Valid and invalid tricolorings of the five-row pyramid.



Colors to Numbers

In truth, this is an algebra puzzle arising from the simplest linear equation in three variables, namely $a + b + c = 0$. Suppose that our number system is not the real numbers, but $\mathbb{F}_3$, the field of integers modulo 3. In $\mathbb{F}_3$, the only numbers are 0, 1, and 2. To add/multiply numbers in $\mathbb{F}_3$, we add/multiply as usual but reduce the result to its remainder upon dividing by 3; table 1 shows the addition and multiplication tables.

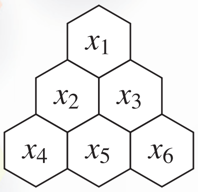
In this number system, there are nine solutions to the equation $a + b + c = 0$: three where a, b, and c all take the same value in $\mathbb{F}_3$, and six where a, b, and c take all three different values in $\mathbb{F}_3$. Thus, this equation perfectly encodes the "all three the same or all three different" condition of the puzzle. We simply

Table 1. Addition and multiplication modulo 3.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Figure 2. Variable assignment for three-row pyramid.



replace colors with integers modulo 3—any assignment of the three colors to the three numbers 0, 1, and 2 will do. Here, we use 0 for red, 1 for green, and 2 for yellow.

Now, we can view the coloring rule for the pyramid as a system of linear equations over $\mathbb{F}_3$. For example, a three-row pyramid can have its cells labeled with variables as in figure 2.

Using the six variables, the coloring condition becomes the following system of three linear equations, which can be encoded into the matrix A shown to the right:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 + x_4 + x_5 &= 0 \\ x_3 + x_5 + x_6 &= 0 \end{aligned} \qquad A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Each column of A represents a variable and each row represents an equation. The entries are given by the coefficient of the corresponding variable in the appropriate equation.

In general, the coloring condition for a pyramid with n rows translates to a system of $n(n-1)/2$ linear equations, each with three variables summing to zero in $\mathbb{F}_3$. The solutions to the linear system are precisely the valid colorings of the pyramid. This is an elegant representation of the problem, and it allows us to study colorings systematically using the techniques of linear algebra. For example, beginning with the matrix A for the system of equations above, there is a standard linear algebra algorithm that produces the matrix

$$N_3 = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The matrix N_3 has interesting properties related to the coloring of the three-row pyramid.

1. Each row of N_3 yields a valid coloring of the three-row pyramid. For example, we interpret row 1 in N_3, which is [1,0,2,0,0,1], as $x_1 = 1$, $x_2 = 0$, $x_3 = 2$, $x_4 = 0$ $x_5 = 0$ and $x_6 = 1$. Figure 3 depicts the corresponding coloring.
2. Every valid coloring of the three-row pyramid can be obtained by adding rows of N_3 together, with repetition allowed. Row addition is performed component-wise, adding numbers in the same columns (modulo 3, as usual). For example, row 2 plus row 3 plus row 3—more compactly, we will write $R_2 + 2R_3$ —results in [1,0,2,2,1,0], shown in figure 4. (Try it yourself: what combination of rows makes the all-green coloring?)
3. The rows of N_3 are *independent* of one another in a strong sense. Not only are all the rows different, but no row can be written as a sum of other rows. You can prove this by considering the entries in the last three columns.

Figure 3. Row 1 of N_3 as a three-row pyramid coloring.

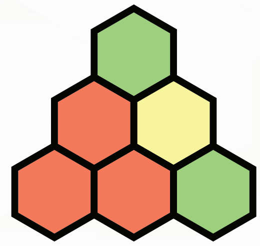
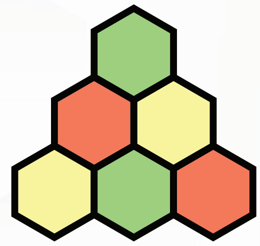


Figure 4. $R_2 + 2R_3$ from N_3 as a coloring.



We never need to use a row more than twice in a combination because we use arithmetic modulo 3. For that reason, solutions to the three-row puzzle can be represented in a meaningful way by ordered triples of numbers from $\mathbb{F}_3$—for example, we use (0,1,2) to represent the combination $0R_1 + 1R_2 + 2R_3$ and (2,2,1) to represent the combination $2R_1 + 2R_2 + R_3$. The all-red (all-zero) coloring corresponds to (0,0,0). With a computer, or even by hand, we could step through all of the $3^3 = 27$ triples, from (0,0,0) to (2,2,2), and produce the corresponding colorings by adding the appropriate rows of N_3.

In a linear algebra class, we'd say that the solutions to the three-row puzzle form a three-dimensional vector space over $\mathbb{F}_3$, and the three special properties of N_3 amount to saying that the rows of this matrix form a *basis* for that space. The solution set may also be called the *null space* associated with the system of equations, and in fact, I used the NullSpace command in *Mathematica* to compute N_3, although it's straightforward to compute by hand. Almost any computer algebra system has a command for this job.

For a pyramid with n rows, applying NullSpace to the appropriate system of equations produces a matrix with n rows and one column for each cell in the pyramid. Each combination of rows added together will yield a valid coloring, and every valid coloring can be produced by adding together various combinations of the rows, from (0,0,…,0) to (2,2,…,2). Therefore, the pyramid with n rows will have 3^n valid colorings; can you find a simpler, more direct proof of this fact?

How Many Clues, and Which Ones?

In the world of Sudoku, many CPU hours have been expended to find the minimum number of clues that determine a unique solution. For pyramid puzzles, the situation is simpler, and one linear algebraic consequence is the following.

Fact. It takes at least n clues to determine a unique solution for an n-row pyramid.

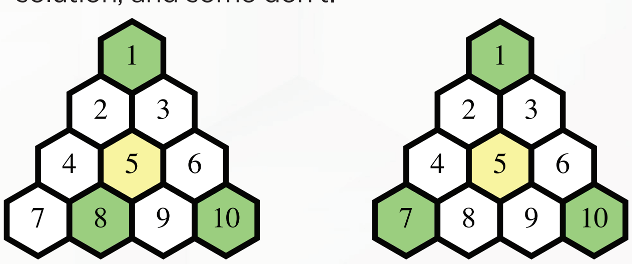
However, not every set of n clues will determine a unique solution for the pyramid with n rows. The clue set on the left in figure 5 determines a unique solution, but the set on the right does not. Try it and see! The notion of independence allows us to algebraically detect the difference between the clue sets {1,5,8,10} and {1,5,7,10}. Using the NullSpace command on the four-row pyramid system of equations produces the matrix

$$N_4 = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We construct two smaller matrices by picking out only those columns of N_4 that correspond to the cells in the clue sets from figure 4: P_1 uses columns 1, 5, 8, and 10 from N_4, while P_2 uses columns 1, 5, 7, and 10.

$$P_1 = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

Figure 5. Some clues determine a unique solution, and some don't.



Remember, P_1 corresponds to the "good" clue set that determines a unique solution, and you can see that its rows have that wonderful independence property: no row of P_1 is equal to any other row, or even to any sum of other rows. On the other hand, P_2 lacks the independence property, as rows 2 and 3 are identical.

We won't prove it here, but this process is exactly how to test whether a set of clue cells is a "good" set that determines a unique solution: select the columns from the NullSpace matrix that correspond to your clue cells, and delete the rest. If the shortened rows are independent of one another, the clue set is good, but if any row can be obtained from the sum of the other rows, then the clue set will allow more than one solution.

Unexpected Relations

You probably realize now that solving a puzzle can be reduced to solving a system of linear equations, which a computer can do in an eyeblink. This does not really reflect how humans solve these puzzles in practice. There are computer algorithms that are guaranteed to solve any Sudoku puzzle, but they are not practical or fun algorithms for humans. The same is true for tricolor pyramids, so let's say a little about how to design puzzles so that they have more human interest to them.

If you examine any correctly colored pyramid with four rows, you will find that the three corners of the pyramid always form a trio: all three are the same, or all three are different, regardless of how the other cells may be colored. We can explain this algebraically by assigning variables to the cells, as shown in figure 6. Working from the bottom up (remembering that addition and multiplication are performed modulo 3, so that 1 + 2 = 0), we see

$$x_4 = 2x_7 + 2x_8,\ x_5 = 2x_8 + 2x_9, \text{and } x_6 = 2x_9 + 2x_{10}$$

which forces

$$x_2 = x_7 + 2x_8 + x_9 \text{ and } x_3 = x_8 + 2x_9 + x_{10}$$

so that, finally,

$$x_1 = 2x_7 + 2x_{10},$$

or equivalently, $x_1 + x_7 + x_{10} = 0$. This means that knowledge of any two of the corners in a four-row pyramid allows the puzzler to deduce the last; the same reasoning applies to any four-row subpyramid in a larger puzzle. In warm-up

Figure 6. The "three corners" pattern.

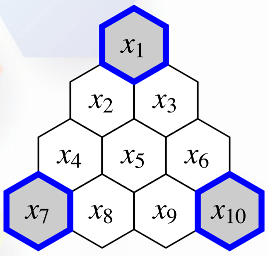
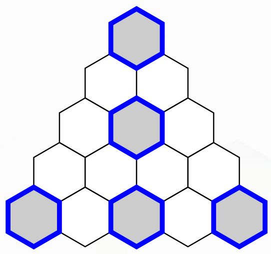


Figure 7. The "five-T" pattern.



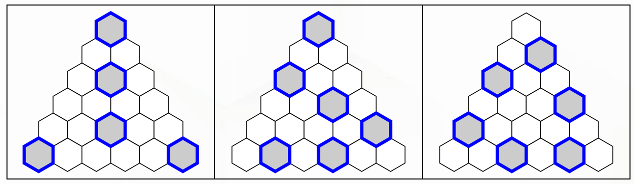
puzzles #3 and #4 from the cover, the "three corners" pattern can be used to deduce a cell with no guesswork. In puzzle #3, the rest of the puzzle unravels quickly; puzzle #4 doesn't yield quite so easily.

Figure 7 shows five cells that also satisfy a simple relation among themselves, regardless of the rest of the pyramid. You'll have to deduce the relation yourself this time. You can also rotate figure 7 to find other sets of five cells satisfying identical relations. See if you can spot how to apply this pattern to solve the rest of puzzle #4 from the cover with no trial and error.

Figure 8 shows a further sampling of patterns that can be used to eliminate guesswork in puzzle solving. Any one of the shaded cells in a pattern can be deduced if you know the others. Keep your eye out for these shapes (and their reflected and rotated forms) as you solve. Or, if you like the challenge of designing puzzles, see how many patterns you can weave into the solution of your puzzle.

Let's consider one final problem. How can you adapt the linear algebraic techniques described in the previous sections to determine if a particular set of cells forms a pattern? You can bet it's all about that independence property.

Figure 8. The "Y", "S", and "O" patterns.



Final Remarks

The tricolor puzzle is a close relative of "Number Pyramid" puzzles commonly used in early grades to practice addition, subtraction, and algebraic thinking. I heartily recommend Jan Hendrick Müller's article, "Exploring Number-Pyramids in the Secondary Schools" (*The Teaching of Mathematics*, VI(3), 2003, pp 37–48; available at *http://www.teaching.math.rs/vol/tm613.pdf*) for a thoughtful and creative look at the mathematics of these puzzles and how they can be used in the classroom.

In the tricolor puzzles, simple patterns like "three corners" and "five-T" emerge as a consequence of reducing the coefficients modulo 3. Consequently, the puzzles acquire a geometric, pattern-spotting element that is not present in ordinary number pyramids. Nonetheless, standard algorithms and theorems from linear algebra can still be used to solve puzzles, test clue sets for solvability, search for useful patterns, and design interesting puzzles around them. Linear algebra turns out to be pretty colorful!

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He would like to thank the students at Fairfield Elementary School, in Fairfield, VA, for their curiosity and questions about hexagonal pyramids, coin-flipping puzzles, geometrical dissections, and other mathematical diversions.

10.1080/10724117.2019.1676117

Solutions to the puzzles on page 2.