

BRAIDED NEAR-GROUP CATEGORIES

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ABSTRACT. A near-group category is an additively semisimple category with a product such that all but one of the simple objects is invertible. We classify braided structures on near-group categories, and give explicit numerical formulas for their associativity and commutativity morphisms.

1. RESULTS

We consider near-group categories; that is, semisimple monoidal categories, whose set of simples consists of a finite group G of invertible elements together with one noninvertible element m . We restrict to the case where m is not a summand in mm , and we will say that G is the underlying group for such a category. Our categories also have a ground ring R , and we assume throughout that R is an integral domain.

Tambara and Yamagami [TY] have classified the possible monoidal structures on such categories. Their result is:

1.1 Theorem (Tambara-Yamagami). *Monoidal near-group categories correspond to pairs (χ, τ) where χ is a nondegenerate, symmetric R -valued bicharacter of G , and τ is a square root of $1/|G|$*

Our main result is the following theorem, which gives necessary and sufficient conditions under which these categories admit a balanced (tortile) braided structure, and parametrizes the distinct braidings.

1.2 Theorem. *Suppose G is a finite abelian group, \mathcal{C} is a monoidal near-group category with underlying group G , and R contains roots of unity of order $8|G|$. Then*

(1) \mathcal{C} admits a braiding if and only if G is an elementary abelian 2-group; that is, every element has order 2.

(2) The nonequivalent braidings on \mathcal{C} are in one-to-one correspondence with $(n+1)$ -tuples $(\delta_1, \dots, \delta_n, \epsilon)$, where $\epsilon = \pm 1$ and $\delta_i = \pm 1$ for all i , and n is the rank of G .

(3) Each braiding of \mathcal{C} has exactly two choices of twist morphisms compatible with it.

Explicit computation of the commutativities further establishes

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1.3 Corollary. *In any braiding of \mathcal{C} ,*

- (1) *the group subcategory generated by invertibles is symmetric commutative.*
- (2) *a basis can be chosen so that every commuting isomorphism is multiplication by an $8|G|$ -th root of unity.*

In section 5 we give a simple example to show that that the order condition of 1.3(2) is sharp. Section 6 contains some additional comments about the conditions on the ground ring R .

1.4 Remarks.

- (1) It is easily seen that if the rank of G is odd then the category does not have an integer-valued dimension function. In particular it is not a category of representations. [TY] shows that some even-rank cases are not representation categories, for more subtle reasons.
- (2) According to [TY], any abelian group can underlie an associative near-group category. Similarly [Q] shows any abelian group underlies a braided group-category. The theorem and corollary show that the structures that extend are quite rare: few associative near-group categories support a braided structure, and few braided group-categories can be embedded in a braided near-group category.
- (3) This rarity may be related to duality. The self-dual elements of a group-category correspond exactly to the elements of order two. Our definition of near-group category requires the object m to be self-dual, and the conclusion is that this forces the invertible elements to be self-dual too. Generalizing the definition to allow two mutually dual non-invertible simple objects might enlarge the number of groups allowed.

2. THE INVARIANTS

2.1 Notation. In a near-group category there are four different kinds of commutes to consider, and we use notation that recognizes their different roles. For $g, h \in G$, the commuting isomorphisms are:

$$\begin{aligned} gh &\rightarrow hg && \text{is multiplication by } \sigma_0(g, h) \\ gm &\rightarrow mg && \text{is multiplication by } \sigma_1(g) \\ mg &\rightarrow gm && \text{is multiplication by } \sigma_2(g) \\ mm &\rightarrow mm && \text{is multiplication by } \sigma_3(g) \text{ on the } g \text{ summand} \end{aligned}$$

2.2 Extracting the invariants. We first want to clarify the result of [TY] cited in 1.1. Saying that χ is a bicharacter of G means that $\chi(ab, c) = \chi(a, c)\chi(b, c)$ and $\chi(a, bc) = \chi(a, b)\chi(a, c)$; this is just a bilinear form written multiplicatively. The technique of [TY] is to establish that for each pair (χ, τ) there is a choice of basis, called a *normal basis*, so that the associativities in the category have the following form: for $a, b, c \in G$,

$$\begin{aligned} \alpha_{a,b,c} &= 1 \\ \alpha_{a,b,m} &= \alpha_{m,a,b} = 1 \\ \alpha_{a,m,b} &= \chi(a, b) \\ \alpha_{a,m,m} &= \alpha_{m,m,a} = \oplus_b 1_b \\ \alpha_{m,a,m} &= \oplus_b \chi(a, b) \\ \alpha_{m,m,m} &= (\tau\chi(a, b)^{-1})_{a,b} \end{aligned}$$

and moreover,

$$\alpha_{m,m,m}^{-1} = (\tau\chi(a, b))_{a,b}$$

Now, to obtain the $(\delta_1, \dots, \delta_n, \epsilon)$ describing the commutativity, fix generators g_1 through g_n for G and choose a normal basis for the parameters χ and τ describing the associative structure. Then the δ_i and ϵ parameters are computed by taking

$$\delta_i = \frac{\sigma_1(g_i)}{\sqrt{\chi(g_i, g_i)}}$$

and

$$\epsilon = \frac{\sigma_3(1)}{\sqrt{\tau \sum_{g \in G} \sigma_1(g)}}$$

It is not obvious that these invariants are ± 1 -valued; this will follow from our examination of the hexagon equations below.

2.3 Realizing the parameters. [TY] gives the construction to realize the parameters χ and τ describing the associative structure. We assume we have that construction in hand and fix a normal basis for it. Realizing the braiding parameters $(\delta_1, \dots, \delta_n, \epsilon)$ means explicitly giving the values of the σ functions.

1. Commuting group elements. Set

$$\sigma_0(g, h) = \chi(g, h)$$

2. Commuting a group element with the noninvertible. First, for the generators g_i , set

$$\sigma_1(g_i) = \sigma_2(g_i) = \delta_i \sqrt{\chi(g_i, g_i)}$$

Any other element g in the group can be written uniquely as a product of generators with each generator appearing at most once (generators have order 2). Say $g = h_1 h_2 \cdots h_k$ is the expression for g as a product of generators. Set

$$\sigma_1(g) = \sigma_2(g) = \prod_{i=1}^k \left(\sigma_1(h_i) \prod_{i < j \leq k} \chi(h_i, h_j) \right)$$

3. Commuting the noninvertible with itself. This also goes in two steps. First, set

$$\sigma_3(1) = \epsilon \sqrt{\tau \sum_{g \in G} \sigma_1(g)}$$

Then, for any other $g \in G$, set

$$\sigma_3(g) = \sigma_3(1) \sigma_1(g) \chi(g, g)$$

Comparing these definitions to the procedure above for extracting invariants, it is clear that the σ 's we have defined do realize the parameters $(\delta_1, \dots, \delta_n, \epsilon)$.

2.4 Note. We see from the formulas that the commutativity inside the group category is entirely determined by the associativity parameters; all the “new” information concerns the behavior of the noninvertible.

3. PROOFS

3.1 The hexagon restraints. The results of the theorem come directly from the hexagon axioms. If we let a, b, c stand for group elements, there are eight types of three-term products to consider: abc , abm , amb , mab , mma , mam , amm , and mmm . Using a normal basis, we can express associativities in terms of χ and τ , and write out the standard hexagon equations corresponding to the eight three-term products:

$$\begin{aligned}
\sigma_0(a, b)\sigma_0(a, c) &= \sigma_0(a, bc) \\
\sigma_0(a, b) &= \chi(b, a) \\
\sigma_0(a, b)\sigma_1(a) &= \sigma_1(a)\chi(a, b) \\
\sigma_2(ab) &= \sigma_2(b)\chi(a, b)\sigma_2(a) \\
\sigma_2(a)\sigma_3(ba^{-1}) &= \chi(a, b)\sigma_3(b) \\
\sigma_3(a^{-1}b)\sigma_2(a) &= \sigma_3(b)\chi(a, b) \\
\sigma_0(a, a^{-1}b) &= \sigma_1(a)\chi(a, b)\sigma_1(a) \\
\sigma_3(a)\tau\chi(a, b)^{-1}\sigma_3(b) &= \sum_{c \in G} \tau\chi(a, c)^{-1}\tau\sigma_2(c)\chi(c, b)^{-1}
\end{aligned}$$

The hexagon for mmm actually gives the matrix equation $AS_2A = S_3AS_3$, where $A = \alpha_{m,m,m} = (\chi(a, b))_{a,b}$, $S_2 = \oplus_a \sigma_2(a)$ and $S_3 = \oplus_a \sigma_3(a)$. The final equation in the list above corresponds to the (a, b) -th entry of the matrix equation.

There are some obvious redundancies among the hexagon equations; eliminating these and simplifying the remaining ones yields the following reduced set of equations, equivalent to the original eight:

$$\sigma_0(a, b) = \chi(a, b) \tag{1}$$

$$\sigma_2(ab) = \sigma_2(a)\sigma_2(b)\chi(a, b) \tag{2}$$

$$\sigma_2(a)\sigma_3(ba^{-1}) = \chi(a, b)\sigma_3(b) \tag{3}$$

$$\sigma_1(a)^2 = \chi(a^{-1}, a) \tag{4}$$

$$\tau \sum_c \chi(ab, c)^{-1}\sigma_2(c) = \chi(a, b)^{-1}\sigma_3(a)\sigma_3(b) \tag{5}$$

Computing the inverse hexagons amounts to inverting associativities and reversing commutativities, producing:

$$\sigma_0(b, a) = \chi(a, b)^{-1} \tag{6}$$

$$\sigma_1(ab) = \sigma_1(a)\sigma_1(b)\chi(a, b)^{-1} \tag{7}$$

$$\sigma_1(a)\sigma_3(ba^{-1}) = \chi(a, b)^{-1}\sigma_3(b) \tag{8}$$

$$\sigma_2(a)^2 = \chi(a, a) \tag{9}$$

$$\tau \sum_c \chi(ab, c)\sigma_1(c) = \chi(a, b)\sigma_3(a)\sigma_3(b) \tag{10}$$

3.2 Proof of 1.2(1). Equations (1) and (6) together with the fact that χ is symmetric imply that $\chi(a, b) = \chi(a, b)^{-1}$ for every $a, b \in G$; that is, χ is ± 1 -valued. But χ is also nondegenerate. If $|a| = 2n + 1$, then $\chi(a, b)^{2n+1} = \chi(1, b) = 1$ for every b . So $\chi(a, b) = 1$ for every b , contradicting nondegeneracy. If $|a| = 2n$ for $n > 1$ then $a^2 \neq 1$ but $\chi(a^2, b) = 1$ for every b , also contradicting nondegeneracy. This shows that every element in G has order 2, as claimed.

3.3 Note. Since group elements are self-inverse, we will now drop the inverse signs from group elements whenever they appear. Likewise, we replace χ^{-1} with χ wherever it appears.

3.4 Proof of Corollary 1.3(1). For invertibles g and h ,

$$\sigma_0(g, h)\sigma_0(h, g) = \chi(g, h)\chi(h, g) = \chi(g, h)^2 = 1.$$

3.5 Proof of 1.2(2).

1. The invariant $(\delta_1, \dots, \delta_n, \epsilon)$ is well-defined. We have to show that our definition is independent of the choice of normal basis that was made. For any group element a , let a_l and a_r be the basis for $\text{hom}(m, am)$ and $\text{hom}(m, ma)$, respectively, in some normal basis; let a'_l and a'_r play the same role in some other normal basis. With respect to any normal basis (by definition), the association $(ma)m \rightarrow m(am)$ is represented by the identity on the 1 summand. That implies that there is some k in the ground ring so that $a'_l = ka_l$ and $a'_r = ka_r$; it follows that the commuting isomorphism $\sigma_1(a) : am \rightarrow ma$ is represented the same with respect to either normal basis, so setting $\delta_i = \sigma_1(g_i) / \sqrt{\chi(g_i, g_i)}$ is well-defined.

It is clear that $\sigma_3(1)$ doesn't depend on the choice of basis of $\text{hom}(1, mm)$; together with the preceding argument, this is enough to establish that the invariant ϵ is well-defined.

2. Giving $(\delta_1, \dots, \delta_n, \epsilon)$ determines the commutativity. We have already noted that σ_0 is totally determined by the associative structure. Equations (3) and (8) together imply that $\sigma_1 \equiv \sigma_2$. Equations (2) and (7) say that σ_1 and σ_2 are determined by their values on the generators g_1, \dots, g_n for G ; finally, (4) and (9) say that $\sigma_1(g_i) = \sigma_2(g_i)$ is determined, up to a sign, by the value of $\chi(g_i, g_i)$. Therefore, giving the δ_i to specify those signs is enough to determine σ_1 and σ_2 .

Equation (3) says that σ_3 is determined by its value on any one group element. For instance, specifying $\sigma_3(1)$ is enough to describe σ_3 entirely. But evaluating (5) for the special case $a = b = 1$, we get

$$\sigma_3(1)^2 = \tau \sum_{c \in G} \sigma_1(c)$$

So $\sigma_3(1)$ is in fact determined up to a sign, and giving ϵ to specify that sign is enough to determine σ_3 , completing the description of all the commutativities.

3. Every $(\delta_1, \dots, \delta_n, \epsilon)$ can be realized. Define the σ functions according to the construction given in 2.3. We have to show that all the hexagon equations are satisfied. Clearly, equations (1) and (6) are satisfied.

Next, we will verify (2) and (7); since the construction defines σ_1 and σ_2 to be equal, and since we can drop inverses, it suffices to check that (2) is satisfied. For a generator g_i ,

$$\sigma_2(g_i)^2 = \left(\delta_i \sqrt{\chi(g_i, g_i)} \right)^2 = \chi(g_i, g_i)$$

so (2) is satisfied on generators. For any element $g = h_1 h_2 \cdots h_k$ expressed as a product of generators,

$$\begin{aligned} \sigma_2(g)^2 &= \left(\prod_{i=1}^k \left(\sigma_2(h_i) \prod_{i < j \leq k} \chi(h_i, h_j) \right) \right)^2 \\ &= \prod_{i=1}^k \sigma_2(h_i)^2 \quad \text{since } \chi^2 = 1 \\ &= \prod_{i=1}^k \chi(h_i, h_i) \end{aligned}$$

Working from the other end,

$$\begin{aligned} \chi(g, g) &= \chi(h_1 h_2 \cdots h_k, h_1 h_2 \cdots h_k) \\ &= \prod_{1 \leq i, j \leq k} \chi(h_i, h_j) \\ &= \prod_{i=1}^k \chi(h_i, h_i) \quad \text{since } \chi(h_i, h_j) \chi(h_j, h_i) = 1 \text{ when } i \neq j \end{aligned}$$

This establishes that $\sigma_2(g)^2 = \chi(g, g)$ for every $g \in G$, so equations (2) and (7) are satisfied.

To show that (3) and (8) are satisfied, it suffices to check one or the other; we show that (8) holds (remember that we can drop inverses):

$$\begin{aligned} \sigma_1(a) \sigma_3(ab) &= \sigma_1(a) \sigma_3(1) \sigma_1(ab) \chi(ab, ab) \\ &= \sigma_1(a) \sigma_3(1) \sigma_1(a) \sigma_1(b) \chi(a, b) \chi(ab, ab) \\ &= \sigma_1(a)^2 \sigma_3(1) \sigma_1(b) \chi(a, b) \chi(a, a) \chi(b, b) \\ &= \chi(a, a)^2 \chi(a, b) \sigma_3(1) \sigma_1(b) \chi(b, b) \\ &= \chi(a, b) \sigma_3(b) \end{aligned}$$

Finally we check that (10) is satisfied; (5) is identical. This is another computation directly from the definitions:

$$\begin{aligned} \chi(a, b) \sigma_3(a) \sigma_3(b) &= \chi(a, b) \chi(a, a) \chi(b, b) \sigma_1(a) \sigma_1(b) \sigma_3(1)^2 \\ &= \chi(a, b) \chi(a, a) \chi(b, b) \sigma_1(a) \sigma_1(b) \tau \sum_c \sigma_1(c) \\ &= \tau \sum_c \chi(a, a) \chi(b, b) \sigma_1(ab) \sigma_1(c) \\ &= \tau \sum_c \chi(a, a) \chi(b, b) \chi(ab, c) \sigma_1(abc) \\ &= \tau \sum_c \chi(ab, abc) \sigma_1(abc) = \tau \sum_c \chi(ab, c) \sigma_1(c) \end{aligned}$$

which shows that (10) and (5) are satisfied and completes the verification that the construction gives a commutativity satisfying all the hexagon equations.

3.7 Proof of 1.2(3). The balance axiom requires that we produce automorphisms θ_s for each simple object s satisfying $\theta_{rs} = \theta_r \theta_s \sigma(r, s) \sigma(s, r)$. Here, that means that for all $g, h \in G$, we must have

$$\begin{aligned}\theta_{gh} &= \theta_g \theta_h \sigma_0(h, g) \sigma_0(g, h) \\ \theta_m &= \theta_g \theta_m \sigma_1(g) \sigma_2(g) \\ \text{and } \theta_g &= \theta_m^2 \sigma_3(g)^2\end{aligned}$$

We know that commutativity in the group subcategory is symmetric, and with respect to a normal basis we know that $\sigma_1 = \sigma_2$. So we can give a slightly simplified statement of the requirements:

$$\theta_{gh} = \theta_g \theta_h \tag{11}$$

$$\theta_g = \sigma_1(g)^2 \tag{12}$$

$$\theta_g = \theta_m^2 \sigma_3(g)^2 \tag{13}$$

Strictly speaking, equation (12) should be $\theta_g = \sigma_1(g)^{-2}$, but $\sigma_1(g)^2 = \chi(g, g) = \pm 1$, so dropping the inverse is harmless. To show that the category is balanced we define

$$\begin{aligned}\theta_g &:= \sigma_1(g)^2 \\ \text{and } \theta_m &:= \pm 1 / \sigma_3(1)\end{aligned}$$

and check that these satisfy the axioms. Equation (12) is obviously satisfied. We check (11):

$$\theta_{gh} = \sigma_1(gh)^2 = (\sigma_1(g) \sigma_1(h) \chi(g, h))^2 = \sigma_1(g)^2 \sigma_1(h)^2 = \theta_g \theta_h$$

and (13):

$$\begin{aligned}\theta_m^2 \sigma_3(g)^2 &= \frac{1}{\sigma_3(1)^2} (\sigma_3(1) \sigma_1(g) \chi(g, g))^2 \\ &= \sigma_1(g)^2 = \theta_g\end{aligned}$$

This shows that there are two choices of twist morphisms which balance the commutativity; conversely, it is obvious that we had to define the twist morphisms as we did, so these are the only two possibilities, concluding the proof of Theorem 1.

3.8 Proof of Corollary 1.3(2). Assume that the σ functions are defined according to the construction in 2.3. We want to show that they take values in the roots of unity. First, $\sigma_0(g, h) = \chi(g, h)$ so σ_0 is ± 1 -valued. Also, $\sigma_1(g)^2 = \sigma_2(g)^2 = \chi(g, g)$ so σ_1 and σ_2 take values that are at most 4th roots of unity.

Now, recall from 3.1 that the hexagon for mmm can be expressed as the matrix equation $AS_2A = S_3AS_3$. Pass to determinants; $A^2 = I$, so $\det A = \pm 1$, and

$$\begin{aligned}\det A \det S_1 &= \det S_3^2 \\ \pm \prod_g \sigma_1(g) &= \prod_g \sigma_3(g)^2 \\ \pm \prod_g \sigma_1(g) &= \prod_g \sigma_3(1)^2 \sigma_1(g)^2 \chi(g, g)^2 \\ \sigma_3(1)^{2|G|} &= \pm \prod_g \sigma_1(g)^{-1}\end{aligned}$$

so $\sigma_3(1)^{8|G|} = 1$. For any other $g \in G$, $\sigma_3(g) = \sigma_3(1) \sigma_1(g) \chi(g, g)$, so $\sigma_3(g)^{8|G|} = 1$, completing the proof.

5. AN EXAMPLE

If we let $G = \langle g \rangle$ be the cyclic 2-group, we obtain examples showing that the bounds of Corollary 1.3(2) are best possible without additional hypotheses. There is only one nondegenerate bicharacter χ on G ; monoidal structures are given by $\tau = \pm(1/\sqrt{2})$, and braidings are given by pairs (δ, ϵ) of ± 1 's. If we leave the parameters generic, we can compute the interesting commutativities (I stands for a primitive 4th root of 1):

$$\begin{aligned}\sigma_1(1) &= \sigma_2(1) = 1 \\ \sigma_1(g) &= \sigma_2(g) = \delta I \\ \sigma_3(1) &= \epsilon \sqrt{\tau(1 + \delta I)} \\ \sigma_3(g) &= \epsilon \delta I \sqrt{\tau(1 + \delta I)}\end{aligned}$$

In this case, regardless of the parameters, it is easily verified that $\sigma_3(1)$ and $\sigma_3(g)$ are primitive 16th roots of unity.

6. THE GROUND RING

Theorem 1.2 says that if R has enough roots of unity, then every $(\delta_1, \dots, \delta_n, \epsilon)$ can be realized as a braiding over R . In general, if R does not have enough roots of unity, some of these will be impossible to realize. The example from section 5 should be sufficiently cautionary: there, none of the parameter sets can be realized unless the exact hypotheses of theorem 1.2 are satisfied. On the other hand, the explicit formulas we have for the associativities and commutativities show that a lack of roots of unity is essentially the only obstruction to realizing the braidings.

We note that near-group categories are never defined over the integers; this is ruled out just by the monoidal structure, since τ is never integral, yet it necessarily appears in the associativity of the noninvertible. We also point out that the bicharacter χ has an important effect on what roots of unity are required. If $\chi(g, g) = -1$ for any $g \in G$, then at least 4th roots of unity are unavoidable in realizing the braidings (this is because $\sigma_1(g)^2 = \chi(g, g)$). Conversely, if $\chi(g, g) = +1$ for every g , then we can refine the proof of 1.3(2) to show that we need only $4|G|$ -th roots of unity to realize all the braidings.

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