How Many Unicycles on a Wheel?

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We are in the world of graph theory here, not circus acts, and we have something interesting to enumerate; let's have a look at what we'll be counting. On the left of Figure 1 is the wheel graph W_4. It is shown alongside several, though far from all, of its *spanning unicycles*.

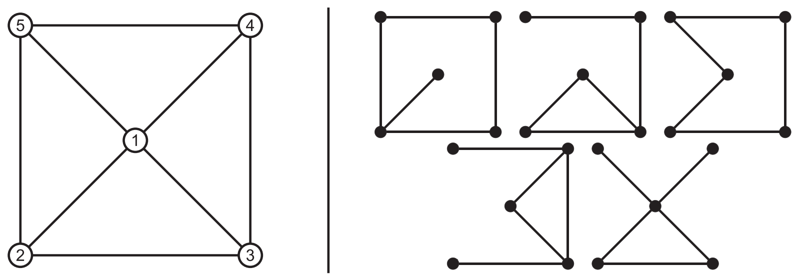

Figure 1 Wheel graph W_4 and a few of its spanning unicycles.

To define our terms: For $n \geq 3$, the *wheel graph* W_n consists of n vertices connected in a cycle (the rim), together with one additional vertex (the hub) which is connected to all of them. The standard drawing of the wheel has the rim vertices equally spaced around a circle with the hub at the center.

A *spanning unicycle* in a graph is a subset of the edges which leaves the vertices connected and contains exactly one cycle. Think of it as using graph edges to build a "ring road" through some or all of the vertices, and then choosing just enough additional edges to reach any remaining vertices off the ring. If you do this on a graph with n vertices, you will find you always use exactly n edges. At this point, you might like to pause and work out how many spanning unicycles can be found in W_4 before proceeding to the following theorem, our main result, which discloses the number of unicycles in any wheel graph.

Theorem. *The wheel graph W_n (on $n+1$ vertices) contains $n \cdot F_{2n-1}$ spanning unicycles, where F_{2n-1} denotes the $(2n-1)$th Fibonacci number.*

So, for example (and to make sure we agree about indices), the wheel graph W_4 in Figure 1 contains $4 \cdot 13 = 52$ spanning unicycles, because we set $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. In the following sections, we shall prove this little, specialized counting theorem about unicycles by applying a bigger, more general counting theorem about trees.

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Kirchhoff and spanning trees

If we remove an edge from the cycle of a spanning unicycle, what remains is a connected graph with *no* cycles; this is known as a *spanning tree* in the original graph. Since the idea of a spanning tree is likely to be more familiar than a unicycle to most readers, we could turn things around and say that a spanning unicycle is just what you get when you add one more edge to a spanning tree (necessarily forming one cycle).

Kirchhoff's matrix-tree theorem, which we will state in a moment, says that the answer to the question, "How many spanning trees are in my graph?" is given by a determinant. In order to state the theorem, we need to know that the *Laplace matrix* of a graph is a square matrix L with one row and one column for each vertex of the graph. The (i, j) entry is given by

$$l_{ij} = \begin{cases} \text{degree of vertex } i, & \text{if } i = j \\ -n, & \text{if there are } n \text{ edges between vertex } i \text{ and vertex } j, \end{cases}$$

where the degree of a vertex is simply the number of edges touching that vertex. For example, the Laplace matrix of our example W_4 is (with the rows and columns ordered according to the labeling of the vertices in Figure 1)

$$\begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{pmatrix}.$$

Remarkably, this simple matrix representation of the graph data reduces the enumeration of spanning trees to a straightforward calculation.

Kirchhoff's matrix-tree theorem. *The number of spanning trees in a graph is equal to the absolute value of the determinant of any minor obtained by deleting any one row and column of the graph's Laplace matrix.*

Thus, according to the theorem, by deleting the first row and column from the matrix above, we can compute that W_4 contains

$$\det \begin{pmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix} = 45 \text{ spanning trees.}$$

Kirchhoff's theorem is a gem, and if you haven't encountered it before, I recommend drawing a few small graphs with their spanning trees, and working out the appropriate determinant to confirm your count. You can find an insightful proof of the theorem in Matoušek's book [**9**]. An article by Benjamin and Cameron [**1**] also presents it along with other applications of determinants to enumeration. In what follows, it will be useful to keep in mind that the theorem holds true even in graphs that may have multiple edges between two vertices.

Although the problem of counting unicycles seems to be only "one edge away" from the problem of counting spanning trees, there is no simple analog of the matrix-tree theorem for unicycles. However, we shall be optimistic and examine the relationship between the two counting problems.

Given a graph G with a specified cycle C, let us define a "collapsed" graph G/C which replaces all of C with a single new vertex, c. The idea is conveyed in Figures 2 and 3. The edges of G between vertices of C vanish in G/C. Any edge from a vertex v outside C to a vertex in C becomes an edge from v to c in G/C. This may result in multiple edges from v to c, as seen in Figure 2.

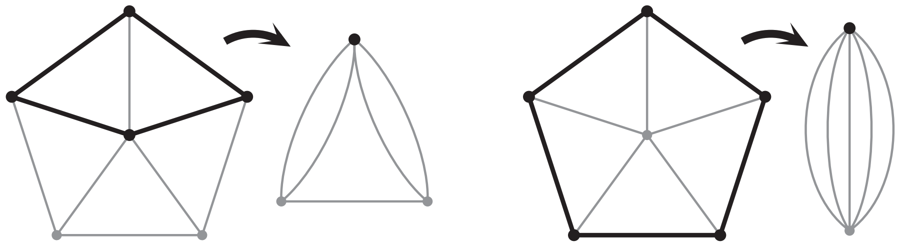

Figure 2 Collapsing different cycles in W_5.

Suppose we have drawn a spanning unicycle on G with cycle C. Collapsing C makes the cycle disappear, leaving a spanning tree in G/C. On the other hand, any spanning tree in G/C can be turned into a spanning unicycle in G with cycle C just by adding the edges of C. These two functions, from unicycles in G to trees in G/C and back again, are inverses of one another, and so we have a bijection.

Lemma 1. *The number of spanning unicycles in G with cycle C is equal to the number of spanning trees in the collapsed graph G/C.*

In principle, this gives a strategy for counting spanning unicycles in a graph G: For each cycle C, use the matrix-tree theorem to count spanning trees in G/C, and add up the results. In practice, this strategy is only as good as our ability to enumerate the cycles in G and organize the sum. Fortunately, in wheel graphs, the cycles are simple and the high amount of symmetry will make the summation easy.

Collapsing and counting

Consider a spanning unicycle on the $(n+1)$-vertex wheel W_n, where $n \geq 3$. Its cycle must be either:

(a) the rim, or

(b) Some k (for $2 \leq k \leq n$) consecutive vertices on the rim, together with the hub, which we call a (hub+k)-cycle.

In case (a), there are simply n ways to complete the unicycle by choosing a spoke to connect the hub to the rim.

In case (b), for any given k, there are n of these (hub+k)-cycles. Since they are all related by rotational symmetries of W_n in its standard drawing, the collapsed graph G/C does not depend (up to isomorphism) on which of them we call C and collapse.

Figure 3 shows the collapse of a (hub+2)-cycle in W_6. The Laplace matrix of the collapsed graph is

$$\begin{pmatrix} \times & \times & \times & \times & \times \\ \times & 3 & -1 & 0 & 0 \\ \times & -1 & 3 & -1 & 0 \\ \times & 0 & -1 & 3 & -1 \\ \times & 0 & 0 & -1 & 3 \end{pmatrix}.$$

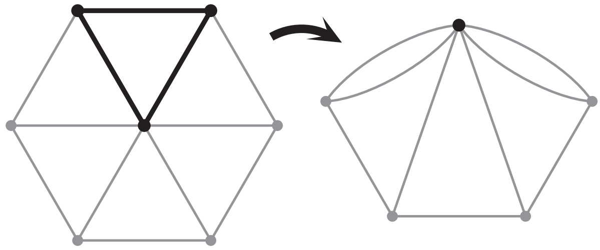

Figure 3 Collapsing any (hub+2)-cycle in W_6.

The $\times$'s represent entries corresponding to the new vertex c, which will be irrelevant as we will discard that row and column to apply the matrix-tree theorem. Now, this particular example nicely illustrates the general case: If we collapse a (hub+k)-cycle in W_n, order the remaining vertices consecutively around the rim, and ignore the row and column of the collapsed vertex, we get an $(n-k) \times (n-k)$ minor in the Laplace matrix with 3's on the main diagonal and -1's above and below it. To apply the matrix-tree theorem, we will need to evaluate the determinant of a matrix in this form.

Let M_k denote the $k \times k$ tridiagonal matrix with 3's on the diagonal and -1's on the sub- and superdiagonal, where

$$M_1 = \begin{pmatrix} 3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}, \quad \text{and so on.}$$

This is where the Fibonacci numbers enter. We see that $\det M_1 = 3 = F_4$ and $\det M_2 = 8 = F_6$, and the pattern continues, as described in the following lemma.

Lemma 2. *For all* $k \geq 1$, $\det M_k = F_{2k+2}$.

This result is a special case of more general results about determinants of tridiagonal matrices [2, 10]. A short paper by Fielder [4] on the subject of enumerating trees also includes this particular case. Still, we can outline a short proof as follows: by expanding the determinant of M_k along the first row, we see that

$$\begin{aligned} \det M_k &= 3 \det M_{k-1} - (-1) \det \left(\begin{array}{c|ccc} -1 & -1 & 0 & \cdots \\ \hline 0 & & & \\ \vdots & & (M_{k-2}) & \\ 0 & & & \end{array} \right) \\ &= 3 \det M_{k-1} - \det M_{k-2} \quad \text{for each } k \geq 3, \end{aligned}$$

and the even-index Fibonacci numbers satisfy the same recurrence (this follows quickly from the definition of the Fibonacci sequence). Since $\det M_1 = F_4$ and $\det M_2 = F_6$ and the two sequences satisfy the same second-order recurrence relation, they agree for all k.

The following lemma is a routine mathematical induction exercise using the defining relation of the Fibonacci sequence.

Lemma 3. *For all* $n \geq 1$, $\displaystyle\sum_{j=1}^{n} F_{2j} = F_{2n+1} - 1$.

With that, we have all we need to establish our main result.

Proof of the theorem. By Lemma 1 and the matrix-tree theorem, $\det M_{n-k}$ represents the number of spanning unicycles in W_n that contain a particular (hub+k)-cycle. Adding the n rim-unicycles to all the (hub+k)-unicycles, we find the total number of unicycles in W_n is

$$n + \sum_{k=2}^{n} n \det M_{n-k} = n + n \sum_{j=0}^{n-2} \det M_j = n + n \sum_{j=0}^{n-2} F_{2j+2}$$
$$= n\left(1 + \sum_{j=1}^{n-1} F_{2j}\right) = n \cdot F_{2n-1}.$$

Lemma 2 replaces the determinant with a Fibonacci number in the second equality, and Lemma 3 provides the evaluation of the sum in the final equality. ∎

Go further with unicycles

The sequence $\{n \cdot F_{2n-1}\}$ which we have obtained here appears as A117202 in the On-Line Encyclopedia of Integer Sequences [**11**], where a different combinatorial connection to wheel graphs is noted (without proof). It is curious that the same numbers also count certain *acyclic* subgraphs of W_n.

The reader who wants practice applying the matrix-tree theorem may take it as an exercise to find the number of spanning trees in W_n. The answer to this is also found in the OEIS, as sequence A004146 [**11**] (you will find connections to many other problems there as well). There are further examples of spanning tree calculations for interesting graph families in the articles of Haghighi and Bibak [**7**] and Hilton [**8**].

Finally, it is worth mentioning that the number of spanning unicycles (or spanning trees) in a graph can be obtained by an appropriate evaluation of the graph's Tutte polynomial. See Bollobás [**3**] for a good introduction. Bollobás mentions the application to spanning trees; the application to unicycles is noted in [**5**] (in the comments following Theorem 7).

Computing the entire Tutte polynomial and then evaluating it is overkill; you can use the deletion and contraction operations which define it to write a simpler recursive formula that is specialized to the unicycle-counting problem. However, dedicated software exists for the purpose of computing Tutte polynomials [**6**], so that is one way to explore the unicycle counting problem experimentally and make discoveries and conjectures. Explore an interesting graph family (for example, the prism graphs or Möbius ladders), and you will likely find an interesting sequence counting the unicycles.

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Summary. The enumeration of spanning trees in a graph is simply accomplished by a determinant (and a great theorem). But what happens when you add one more edge to a spanning tree? The resulting "unicycle" structures in a graph are harder to count, but we explore the problem in the family of wheel graphs, where the enumeration leads to a tidy answer and some old familiar friends.

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