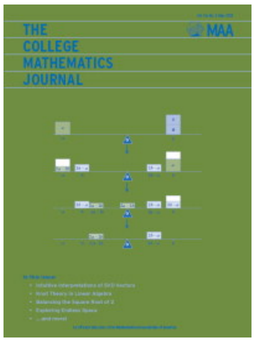

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	a	b	c	d	e
1					1
2					1
3	1				1
4		1		1	
5	1				

Figure 1. A miniature port-and-sweep problem. Play to a single counter at c4.

Port-and-sweep solitaire is a modern offshoot of the centuries-old family of peg solitaire puzzles. The rules were introduced in *Math Horizons* [6] in 2010, along with some mathematical analysis and a few sample problems. However, no single problem in the *Math Horizons* article seemed as “iconic” as the best-known peg solitaire problems, nor as challenging.

In the present article, we offer some more substantial and attractive port-and-sweep problems which feel comparable to the challenge and appeal of the best problems in peg solitaire.

Although this article is meant to be self-contained, readers may find it easier to absorb the rules of the puzzle with an interactive tutorial. Such a tutorial is available at the author’s homepage [5]. The problems from this article are playable there, as well as many others.

The Rules

Port-and-sweep is played with counters on a board with a square grid. The fundamental rule of port-and-sweep is:

- **Each square may hold up to 2 counters – no more.**

Thus, the state of each square on the board can be represented by a 0, a 1, or a 2. In the figures, we will leave squares empty when they contain no counters.

There are two kinds of moves, which you can play in any direction (left, right, up or down):

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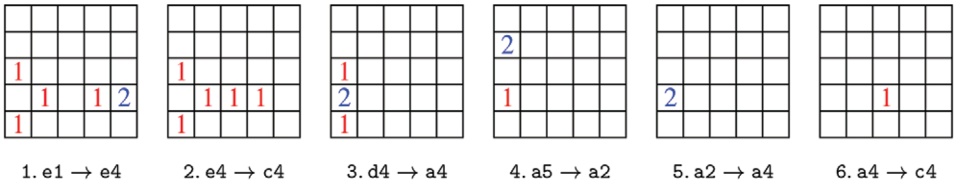


Figure 2. Solution steps for the problem from Figure 1.

- **Port move:** Add $(-2,\ 0,\ +1)$ to three consecutive squares on the board. In other words, take away two counters from a square and add one to a square two steps away, horizontally or vertically. The in-between square is unaffected. Due to the fundamental rule, a port move must begin at a 2 and end at a 0 or 1.
- **Sweep move:** Add $(-1,\ -1,\ -1,\ \ +2)$ to four consecutive squares on the board. In other words, take one counter each from three nonzero squares in a row, then add two to a square at the end of the row. A sweep move must end at a 0.

Each move leaves one less counter on the board. The challenge in most port-and-sweep problems is to reduce a given arrangement of counters to a simpler one—typically, with just one counter remaining—by means of port and sweep moves.

Why these particular moves? As we will see, they are not arbitrary; they were chosen according to an algebraic principle. In terms of practical game play, though, the sweep move can be seen as a direct extension of the one allowable move in peg solitaire, which adds $(-1,\ -1,\ +1)$ to three consecutive cells. By itself, the sweep move is too clumsy to allow interesting puzzles. Adding the port move, however, opens up a rich field of possibilities.

As an example, Figure 1 shows an arrangement of seven counters on a 5×5 board. It can be reduced to a single counter by the following six moves (with reference to the row and column labels shown in Figure 1): (1) Sweep e1 → e4; (2) Port e4 → c4; (3) Sweep d4 → a4; (4) Sweep a5 → a2; (5) Port a2 → a4; (6) Port a4 → c4. The steps are illustrated in Figure 2.

The 5×5 square is the smallest board that allows much interesting gameplay. Problems of a dozen or so moves on this board can make pleasant brainteasers. As a small sample of the possibilities, Figure 3 presents three problems which can be reduced to a single counter in the center square.

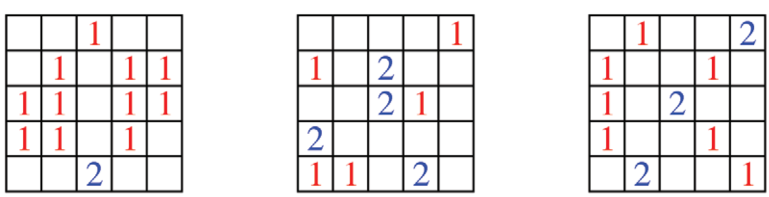


Figure 3. Three problems that play to a single counter in the center.

Small problems with sparsely-placed counters, such as those in Figure 3, can be challenging to solve and to design, just as in peg solitaire. However, most well-known problems in peg solitaire are of a different type: they begin with pegs distributed uniformly over a nearly full board (typically, with just one space left open). The exact shape of the board is important if such problems are to be solvable [2]. Let us now introduce several boards which allow analogous problems for port-and-sweep solitaire. After presenting the problems, we will discuss the relevant mathematical theory in more detail.

Three families of problems

The English Board (seven problems) The best-known peg solitaire board is the symmetrical, cross-shaped board of 33 squares which is known in the literature as the English board. Bell [2] explains the unique properties which characterize this board.

Figure 4 shows a port-and-sweep problem on the English board, which can be played, in 33 moves, to a single counter in the center of the board. It should prove challenging, even for those who are experienced with peg solitaire. It is unlikely that one will stumble onto a solution by heedless play.

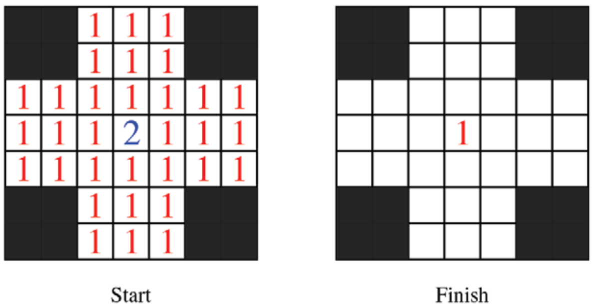


Figure 4. Port-and-sweep analog of an iconic peg solitaire problem.

We will be introducing more puzzles with features in common with this one, and some terminology is useful in describing them.

First, note that the net effect of solving the problem in Figure 4 is to subtract 1 from each square on the board. Any problem whose solution reduces the count in each square by 1 will be called a *subtraction problem*.

For a subtraction problem to end with a single counter, it must begin with a 2 in one square and 1s in every other square. For the rest of this article, we will assume that "subtraction problem" refers to a problem of this type—beginning with a single 2 in a sea of 1s, and ending with a single counter.

One of the distinguishing characteristics of the English board as it pertains to peg solitaire is that it is "universally solvable"—that is, if one begins with a single empty square on the board, then one can finish with a single peg remaining in that originally-empty square [2]. It is a remarkable coincidence that the same board is equally kind to port-and-sweep puzzlers.

Proposition 1. *Every subtraction problem on the 33-square English board is solvable.*

In other words, if you put one counter in each square of the English board, and add one more counter to a square of your choice, then you'll be able to finish with the board empty except for a single counter in that same chosen square. Up to symmetry, there are seven distinct subtraction problems on the English board, and each has its individual flavor. The “central problem” shown in Figure 4 is intermediate in difficulty among them.

No proof of this proposition is provided. I leave it to the reader to discover solutions to these problems.

The Swiss Board (nine problems) Figure 5 poses a subtraction problem on a board of forty-five squares, with four gaps in the board (indicated by darkened squares). The gaps are not part of the board; they may not contain counters. However, port moves across the gaps are allowed. I have dubbed this the “Swiss” board, not for any genuine national association, but merely for the presence of the swiss cheese-like holes.

Gaps are not a common feature of peg solitaire boards, although Bell [2] does provide a pair of examples with nice solvability properties. Gaps can be an interesting feature in port-and-sweep, though, as they provide an obstacle to sweep moves in certain areas of the board, while still allowing ports. Up to symmetry, there are nine different subtraction problems possible on the Swiss board. Like the English board, the Swiss provides bountifully.

Proposition 2. *Every subtraction problem on the 45-square Swiss board is solvable.*

The “central” problem shown in Figure 5 is among the easiest of the nine. All the solutions, again, are left to the reader to discover.

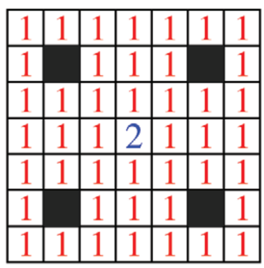


Figure 5. One of nine subtraction problems on the Swiss board.

The German Board (five problems) The problem in Figure 6 uses a 32-square board formed by deleting the corners of a 6×6 square. I have dubbed this the “German” board (an arbitrary name, just continuing the theme of nationalities). It lacks a central square but has a “universal solvability” property, just like the English and Swiss boards.

Proposition 3. *Every subtraction problem on the 32-square German board is solvable.*

Up to symmetry, there are five such problems. They are all of approximately equal difficulty and much easier than the English and Swiss problems. This is mostly due to the lack of tight spaces on the German board. They make a nice warm-up for the harder problems. On this board, it is possible to stumble onto solutions without a great deal of careful planning.

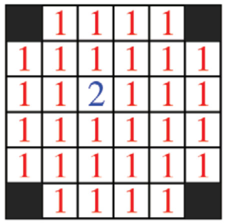


Figure 6. One of five subtraction problems on the German board.

Necessary conditions for solvability

The fact that subtraction problems are solvable on the English, Swiss, and German boards is a special feature; this is not the case for all boards. There is a system of algebraic invariants which presents an obstruction to solvability in many cases.

It will be convenient to use the field of nine elements, denoted $\mathbb{F}_9$. This field consists of numbers of the form $a + bi$, where a and b are elements of the integers mod 3, and $i^2 = 2$. Concretely,

$$\mathbb{F}_9 = \{0, 1, 2, i, 1+i, 2+i, 2i, 1+2i, 2+2i\}.$$

For the purpose of defining the invariants, we will coordinatize the squares on our boards with integer coordinates in the usual Cartesian manner, letting the lower left corner be $(0, 0)$. Let S denote a game position (that is, an arrangement of counters on the squares), and let $S(a, b)$ denote the number of counters on square (a, b). Define

$$\pi(S) = \sum_{(a,b)} S(a, b) i^{a+b} \quad \text{and} \quad \mu(S) = \sum_{(a,b)} S(a, b) i^{a-b}.$$

Thus defined, π and μ are $\mathbb{F}_9$-valued functions. The following proposition is proven in the *Math Horizons* article [6], but the proof is a straightforward calculation which can be taken as an exercise.

Proposition 4. *If a position S' is obtained from position S by any port or sweep move, then $\pi(S') = \pi(S)$ and $\mu(S') = \mu(S)$.*

In other words, the π and μ functions are invariant under legal moves. They are analogous to the $\mathbb{F}_4$-valued functions A and B defined by de Bruijn [4] for ordinary peg solitaire positions. In general, port-and-sweep has a relationship with mod 3 arithmetic that is analogous to ordinary peg solitaire's relationship with mod 2 arithmetic, and this is by design. The port and sweep moves were specifically chosen so that π and μ as defined here would be invariant. (They were also chosen as the simplest set of moves which would allow interesting puzzles. There are other moves which preserve π and μ, and the reader may enjoy looking for examples.)

Let B denote a board—that is, simply, a set of squares, such as the English, Swiss, or German boards. Let B_1 denote the game position with one counter on each square of B.

Definition 1. *B is a null-class board if $\pi(B_1) = \mu(B_1) = 0$.*

The sums to evaluate $\pi(B_1)$ and $\mu(B_1)$ by definition may seem tedious, but note that any pair of squares "a port away" from each other—i.e., agreeing in one coordinate and differing by two in the other coordinate—will cancel each other out. By crossing out canceling pairs, you can reduce the calculation to a sum involving just a few squares (possibly none at all). In this way, if you care to check, you can confirm that the English, Swiss, and German boards are all null-class, but the 5×5 square board in Figure 1 is not.

Proposition 5. *If some subtraction problem on the board B is solvable, then B is null-class.*

Proof. Suppose that position S' is obtained from position S by a sequence of legal moves that, ultimately, subtracts one from each square on board B. Then

$$\begin{aligned}\pi\left(S'\right) &= \sum_{(a,b)\in B} S'(a,b)i^{a+b} \\ &= \sum_{(a,b)\in B} \left(S(a,b)-1\right) i^{a+b} \\ &= \pi\left(S\right) - \pi\left(B_1\right) \\ &= \pi\left(S'\right) - \pi\left(B_1\right),\end{aligned}$$

with the last line following from the invariance of π under legal moves. This implies that $\pi\left(B_1\right) = 0$, and the same argument applies to μ as well. Thus, B is null-class. ■

In other words, "null-class" is a necessary condition for the solvability of subtraction problems on a particular board, which is very easy to check. Unsurprisingly, however, it is not sufficient to guarantee the solvability of all subtraction problems. We shall soon see an example.

Note that, since the calculation of π and μ takes place in $\mathbb{F}_9$, the conclusion applies even if S' is obtained from S by subtracting 1 from each square mod 3—that is, S' may have twos in squares where S has zeros. You might also note that the same argument would apply to any sequence of moves that ultimately *adds* one (mod 3) to each square of the board.

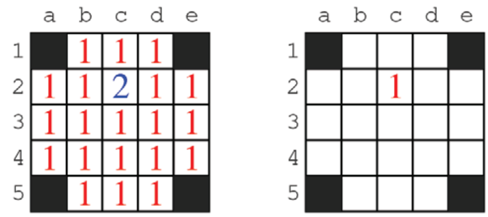
That raises the possibility of *addition problems* which add one (mod 3) to every square on the board (changing 0 to 1, 1 to 2, and 2 to 0). An addition problem which ends with a single counter must begin with a single empty square and a 2 in every other square. Anecdotally, these problems (when solvable) are simply less fun than subtraction problems. They tend to be long (because of the large number of counters at the start), but dull; it is easy to stumble onto solutions without much thought.

Impossibilities

Up to symmetry, there are five subtraction problems on the 21-square board in Figure 7. You can easily check that the board is null-class, so the π and μ invariants do not show any obstruction to solving subtraction problems on this board. However, if you begin with a 2 in the center, you will find very quickly that the problem cannot be solved (you can play a maximum of four moves).

The problem in Figure 7 may seem more plausible; you can play it down to a much smaller number of counters. However, it cannot be reduced to a single counter. Starting with the 2 in other positions doesn't help.

Proposition 6. *None of the subtraction problems on the board of Figure 7 are solvable.*



Start Finish

Figure 7. A plausible, but impossible subtraction problem on a 21-square null-class board.

Proof. By symmetry, we may assume that the initial 2 in the problem is located at `c3`, `c2`, `c1`, `b2`, or `b1`. For any position S on this board, define

$$\text{Count}(S) = \sum_{(a,b)} S(a,b) \cdot V(a,b),$$

where V is as shown in Figure 8, and the sum is evaluated in the ordinary integers (*not* mod 3). While V could be interpreted as a game position, we will refer to it as a "vector," and Count(S) should be thought of as the dot product of S with V over the integers. The Count values for the starting and finishing positions of the various subtraction problems are as follows:

Initial 2 Position	Initial Count	Final Count
`c3`	8	2
`c1`	7	1
`c2`, `b2`, `b1`	6	0

In each case, the Count value would have to decrease by exactly 6 during the course of a supposed solution. The changes caused by legal moves are as follows; in particular, Count is nonincreasing under legal moves.

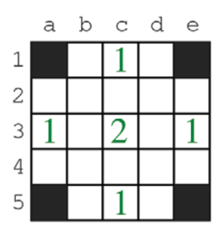


Figure 8. "Count vector" V on the 21-square board.

Move	Change in Count
Port `c3` → `c1`, `c3` → `c5`, `c3` → `a3`, or `c3` → `e3`	−3
Sweep `c1` → `c4`, `e3` → `b3`, `c5` → `c2`, or `a3` → `d3`	−3
All others	0

In any of the problems under consideration, cells `a3`, `e3`, and `c5` initially contain 1's. At some point in a solution, each of those cells has to be reduced to zero. This could happen by a sweep move originating at the cell in question, which would lower the count by 3. It could also happen by a port move originating in the cell; this, by itself, would not affect the Count value, but it would have to be preceded by at least one port *into* the cell, which also lowers the count by 3.

Thus, clearing all three of `a3`, `e3`, and `c5` would lower the count by at least 9, but that is too much; the difference between final and initial count is only 6. No solution is possible. ■

In the peg solitaire literature, nonincreasing functions such as the one defined here by vector V are called *resource counts* [1] or *pagoda functions* [3]. The resource count V used here has the additional property that, although it can decrease, it is invariant mod 3.

Things to try

For the reader who wants more to investigate, I will conclude by posing two proof problems. The first can be done constructively, by devising a suitable algorithm.

Problem 1. *Prove that every subtraction problem on any* $4 \times n$ *rectangle, with* $n \geq 4$*, is solvable.*

You might want to begin by confirming that the $4 \times n$ rectangle is null-class.

Problem 2. *Prove that the 25-square diamond board (Figure 9) is null-class. Then prove that none of the subtraction problems on this board are solvable.*

This second problem can be done by forcibly enumerating the positions on a computer. This is not satisfying, but I haven't found a better way. Perhaps a reader will find a beautiful way to complete the proof without computer assistance.

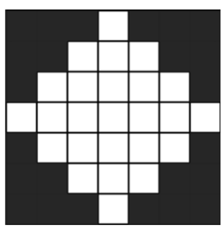


Figure 9. The 25-square diamond board.

Summary. Port-and-Sweep Solitaire (PaSS) is a puzzle game related to the classic pastime of peg solitaire. One of the great surprises of peg solitaire is that its analysis is closely tied to mod 2 arithmetic, and PaSS has an analogous relation to mod 3 arithmetic, which we explain in the article. We give several families of challenging PaSS puzzles for the reader to solve and demonstrate typical techniques for identifying problems which cannot be solved.

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