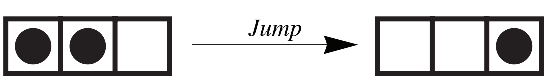
Port-and-Sweep Solitaire

Jacob Siehler

Image courtesy of Cristóbal Vila - etereaestudios.com

How does this happen? I just wanted a nice game where I didn't have to count higher than two, and I ended up dealing with imaginary numbers. But let me back up: I've been a little obsessed with a puzzle lately, and I would like to explain what's puzzling me and how the square root of –1 can sneak in where you least expect it. The puzzle in question is a relative of the classic peg solitaire game pictured above, and before I introduce it properly, we can have a quick brush-up on the traditional version. You could write a whole book on the subject [1] but as you probably know, the game is quite simple: it is played on a grid, where each square may hold a peg (or a marble, or some other marker). The only allowable move is to jump one peg over an adjacent peg into an empty square, removing the jumped peg from the board:



This can be done in any direction on the grid—up, down, left, or right. An equivalent way of looking at the move is that it consists of adding | -1 | -1 | +1 | to the "peg count" of three consecutive squares on the grid—subject to the rule that a square can't contain more than one peg, or a negative number of pegs.

The best-known problem in ordinary peg solitaire (OPS) asks the player to reduce an almost-full board of 33 squares to a single peg in the center square (see photo), but of course other problems are possible—easier problems to warm up on, or fresh challenges for those who have solved the classic configuration [8]. Eventually, though, a feeling of familiarity with the peg solitaire "problem space" sets in, and one abandons the game, abandons solving individual puzzles in favor of proving theorems, or seeks a twist on the rules to make the game new again. Opting for the latter, I propose a variation that I call "Port-and-Sweep Solitaire" (PaSS). PaSS is also played on a grid, and the First Rule of PaSS is that each square may hold 0, 1, or 2 counters—no more, no less. I usually play on my computer, but checkers on a checker-board work just as well (and are pleasingly tactile). PaSS permits two types of move:

1. Sweep move: Add | -1 | -1 | -1 | +2 | to four consecutive squares on the board, and
2. Port move: Add | -2 | 0 | +1 | to three consecutive squares on the board.

Both moves are subject to the First Rule, and as in OPS, they may be performed in all four directions. Lacking several hundred years of tradition, PaSS does not yet boast a single defining problem, but it does admit a wealth of varied and interesting ones. Figure 1 is a small, but typical, puzzle: we're asked to reduce a given arrangement of sixteen counters on a 5×5 board to a single counter in the center. Up to symmetry, there are just three opening moves possible: port a 2 to the center, port a 2 to one of the corners, or sweep three 1's to the edge.

		2		
	1	1	1	
2	1		1	2
	1	1	1	
		2		

		1		

Figure 1. Left: A small puzzle in its initial state. Right: The desired final state.

Figure 2 shows a partial solution to this problem, using the third option to start. See if you can carry it to completion. This is a very forgiving problem; there are many ways to complete the solution I've begun, and my opening isn't too special. Any two moves at the start can be carried to a successful conclusion. But if you arrive at one of the positions in Figure 3 on your third move, you will inevitably get stuck with no legal moves and at least three counters on the board.

Figure 4 offers three more problems on the 5×5 board, which are somewhat less forgiving. In one case you can go

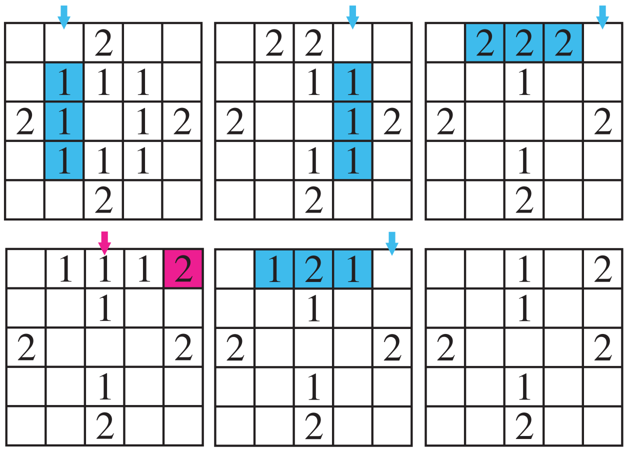


Figure 2. A promising opening sequence of moves.

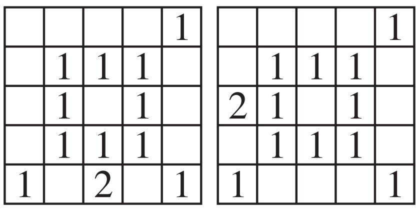


Figure 3. Dead-end configurations.

astray on the first move! Despite the modest length of the problems and the small board size, I think you'll find their solutions satisfying. See Problem 250 in the Playground on page 30.

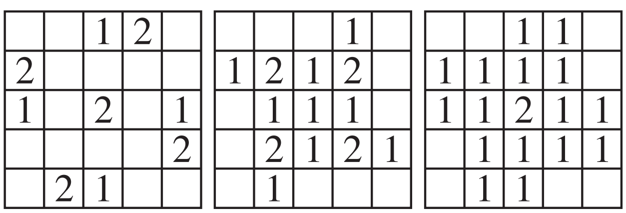


Figure 4. Three more games to try on a 5 × 5 board.

Sorry, You Can't Get There from Here

Trying to reduce the problem in Figure 5 to a single counter will be less satisfying, however, and after a certain number of failed attempts, you will begin to suspect that it can't be done. How can you prove that no solution exists, without examining every possible sequence of moves?

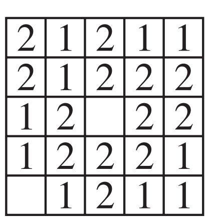


Figure 5. An impossible game.

An elegant way is to use an invariant—a function whose value is determined by the state of the board and will not vary when we change the board by legal moves. If such a function can be found that takes one value on the pristine state of the problem and a different value on the target state, then no sequence of legal moves can ever connect the two. PaSS has a natural affinity for mod 3 arithmetic, and we can take advantage of that to define a valuable pair of invariants for the game. These invariants will assign to each board a value in a charming and petite arithmetic universe of just nine elements; namely, the set

$$F_9 = \{0, 1, 2, i, 1 + i, 2 + i, 2i, 1 + 2i, 2 + 2i\}.$$

F_9 is a discrete, mod 3 analogue of the complex number system. Elements have the form $a + bi$; we can refer to a as the real part and b as the imaginary part. Addition uses the simple rule $(a + bi) + (c + di) = (a + c) + (b + d)i$, reducing both parts mod 3 to stay in the range {0, 1, 2}. To multiply, treat i as a square root of –1, but in the mod 3 world, –1 is the same as +2, so you can avoid minus signs by declaring $i^2 = 2$. That leads to the peculiar, but totally legitimate, multiplication formula

$$(a + bi) + (c + di) = (ac + 2bd) + (ad + bc)i.$$

F_9 is an abelian group under addition, and the nonzero elements form an abelian group under multiplication—that is, F_9 is a field of nine elements. We don't need much of the multiplicative structure for the present purposes, but note that $i(2i) = 1$, that is, $i^{-1} = 2i$.

Now, coordinatize your board in the usual Cartesian manner, letting the lower left square be (0,0). Let $S(a,b)$ denote the number of counters on square (a,b), and define

$$\pi(S) = \sum_{(a,b)} S(a,b)\, i^{a+b} \text{ and } \mu(S) = \sum_{(a,b)} S(a,b)\, i^{a\ \ b},$$

where all arithmetic—sums, products, and powers—takes place in F_9, so each of the sums above, no matter how lengthy, will reduce to one of the nine elements.

Suppose a position S' is derived from S by a sweep to the right, taking one counter from each of (a, b), $(a + 1, b)$, $(a + 2, b)$ and adding two to $(a + 3, b)$. Then

$$\begin{aligned}\pi(S') &= \pi(S) - i^{a+b} - i^{a+b+1} - i^{a+b+2} + 2\, i^{a+b+3} \\ &= \pi(S) - i^{a+b}(1 + i + i^2 + i^3) \\ &= \pi(S),\end{aligned}$$

since the sum in parentheses is zero in F_9. Similar calculations show that the value of π is unchanged by sweeps in the other three directions. And if S' is derived from S by porting a 2 from (a, b) to $(a + 2, b)$, then

$$\begin{aligned}\pi(S') &= \pi(S) - 2\, i^{a+b} + i^{a+b+2} \\ &= \pi(S) + i^{a+b}(1 + i^2) \\ &= \pi(S),\end{aligned}$$

and likewise for ports in other directions. This shows that the value of π remains unchanged under all legal moves in the game, and it's routine to check that μ has the same property. Thus, the values of π and μ separate the possible positions on a given board into 81 distinct classes, and no play between positions in different classes is ever possible. A 5×5 board with a single counter in the center has $\pi = 1$ and $\mu = 1$.

What of the board in Figure 5? Its invariants are $\pi = 2 + i$ and $\mu = 1 + 2i$, so it can never be played to a 1 in the center. In fact, it could never be played to a single 1 or single 2 on any square, since no single term in the sums for π and μ can contribute both a real and imaginary part.

I should admit that I did not have to overexert myself to discover π and μ, because I knew of an analogous pair of invariants for ordinary peg solitaire, described by de Bruijn [6], using mod 2 arithmetic and a field of four elements. PaSS can therefore be seen as a plausible answer to the analogy problem, "What game is to the field of nine elements as peg solitaire is to the field of four elements?"

More Problems: Four in the Corners

Back to problems you can solve: Figure 6 offers a few additional problems on a 6×6 board. Boards with even sides have no center square, so as an attractive, symmetric alternative, these are designed to play down to four 1's, one in each of the four corners.

			1		
		1	1		
1	1	1	1	1	
	1	1	1	1	1
		1	1		
		1			

2				1	2
1		2	1		
	1	1	1	2	
	2	1	1	1	
		1	2		1
2	1				2

1		1	2		1
	1	2	2	1	
2	2			2	1
1	2			2	2
	1	2	2	1	
1		2	1		1

2	1	1			2
	1	1	2	1	1
	2	1	1	1	1
1	1	1	1	2	
1	1	2	1	1	
2			1	1	2

Figure 6. Some four-corners games, where the winning configuration has a 1 in each corner.

How big is the "problem space" of potentially interesting four-corners puzzles? Not very big, if we're picky about what we consider an interesting problem. I prefer problems with a lot of symmetry, and there are $3^9 = 19{,}683$ six-by-six boards that have 90-degree rotational symmetry (including the four-corners configuration itself, and some dull things like the all-zero board). Taking the π and μ invariants into account reduces the possibilities by a factor of 3^4, so at most $3^5 = 729$ of these could plausibly be played to four corners. Among those, a large number share the disappointing property exhibited by Figure 7: one can easily reduce the red cells to a single 1 in the upper left corner, using ports alone, without affecting the rest of the board.

2	2	1	2	2	2
2	2	2	1	2	2
2	1	2	2	2	1
1	2	2	2	1	2
2	2	1	2	2	2
2	2	2	1	2	2

Figure 7. A four-corners game where no sweeping is needed.

Due to symmetry, this can be repeated three more times, verbatim, to solve the problem. Boring! A port-and-sweep problem should require both ports and sweeps, I'm sure you agree. Eliminating the boring problems leaves a list of 273 candidates to be investigated—some that can't be solved, some too small or too obvious to be interesting, some too large to be fun, and just a few in the Goldilocks region: tantalizing, but elusive. Of course, if we're open to problems with less symmetry, there will be many more possibilities, and there are many tactics yet to be discovered which will aid in their construction and solution.

As a parting shot, the 7×7 board in Figure 8 was generated by a computer working backward from a single 1 in the center and trying to obtain a symmetric configuration (in this case, just a reflective symmetry). I did not have the computer save its steps, so I know the problem can be solved, but I don't know how! I believe it's large enough to vex attempts to solve it by sheer computing power—but perhaps a reader will discover a more ingenious, tactical approach. See Problem 250 in the Playground on page 30.

1	2	2	2	1	2	2
2	1	2	2	2	1	2
2	2	2	2	2	2	2
2	2	2	2	2	2	1
1	2	2	2	2	2	2
2	1	2	2	2	1	1
2	2	2	1	2	1	2

Figure 8. Can you play this to a single 1 in the center?

Mutatis Mutandis

Ordinary peg solitaire has been studied extensively, and while Beasley's book [1] is the most comprehensive single work, other stimulating articles can be found. Anyone who is keen to explore will find that most of the problems and techniques that have been developed for OPS can be successfully, ahem, ported over to PaSS—but there are surprises to be found in the process. For example, "resource counts" can be constructed which provide an additional tool for proving certain PaSS problems unsolvable, but they are governed by more restrictive inequalities than in OPS. PaSS seems to be more complicated than OPS on a one-dimensional ($1 \times n$) board. And an investigation of the "Solitaire Army" problem ([2], [4]) will show that PaSS counters have much more forward mobility than their ordinary peg counterparts. That problem might lead you to wonder, "What is to PaSS as the Golden Ratio is to OPS?"—but I don't believe any works of art will be inspired by the answer.

Now, I'm going to have just one more try at the game in Figure 8…

Further Reading

[1] John Beasley, *The Ins and Outs of Peg Solitaire*, Oxford University Press, 1985.

[2] George I. Bell, George Bell's Peg Solitaire Page, `http://home.comcast.net/~gibell/pegsolitaire/index.html`.

[3] George I. Bell, A fresh look at peg solitaire, *Mathematics Magazine*, **80**(1) 2007, 16–28.

[4] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy, *Winning Ways for Your Mathematical Plays*, volume 4, 2nd edition, AK Peters, 2004. See chapter 23, pages 803–842.

[5] Arie Bialostocki, An application of elementary group theory to central solitaire. *The College Mathematics Journal*, **29**(3) 1998, 208–212.

[6] N. G. de Bruijn, A solitaire game and its relation to a finite field, *Journal of Recreational Mathematics*, **5** 1972, 133–137.

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[8] Jacob A. Siehler, Problems in peg solitaire, `http://demonstrations.wolfram.com/ProblemsInPegSolitaire/`.

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