Now, $C_5 \cap H$ has as vertices the 30 permutations of $(1, 1, 0, -1, -1)$. Here the geometric calculation of $\text{Vol}(C_5 \cap H)$ is more difficult. In fact, $C_5 \cap H$ has 10 congruent facets in the 10 hyperplanes $x_i = \pm 1$ $(i = 1, \dots, 5)$. The facet with $x_1 = -1$ has as vertices the 12 permutations of $(1, 1, 0, -1)$ (with the x_1-component omitted) and the center (of gravity of the vertices) $(-1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. It is bounded by 4 regular sixgons with equations $x_i = 1$ $(i = 2, \dots, 5)$ and 4 regular triangles with equations $x_i = -1$ $(i = 2, \dots, 5)$ with side length $\sqrt{2}$. Hence, each facet is a truncated tetrahedron with edge length $\sqrt{2}$ and volume $\frac{23}{3}$. So we can dissect $C_5 \cap H$ into 10 congruent 4-dimensional pyramids with base volume $\frac{23}{3}$ and height $\frac{1}{2}\sqrt{5}$ to get $\text{Vol}(C_5 \cap H) = 10 \cdot \frac{1}{4} \cdot \frac{23}{3} \cdot \frac{1}{2}\sqrt{5} = \frac{115}{12}\sqrt{5}$.

4. OPEN QUESTIONS.

Question 1. Ball's proof of Hensley's conjecture $\text{Vol}(C_n \cap H) \leq 2^{n-1}\sqrt{2}$ is quite complicated. Is there a simple proof using Theorem 2?

Question 2. Ball generalized Theorem 1 to subspaces H_k of arbitrary dimension $k < n$ and used this generalization to prove $\text{Vol}_k(C_n \cap H_k) \leq 2^k \sqrt{2}^{\,n-k}$. This bound is the best possible if $2k \geq n$, but, for example, the maximal area of $C_5 \cap H_2$ is unknown (see [**2**] or [**7**]). Can we generalize Theorem 2 and obtain an elementary formula for $\text{Vol}_k(C_n \cap H_k)$?

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Mathematisches Institut der Universität Koblenz-Landau, Campus Koblenz, D 56070 Koblenz
frank@uni-koblenz.de
riede@uni-koblenz.de

Tiling Hamiltonian Cycles on the 24-Cell

Jacob A. Siehler

Abstract. We present a construction for tiling the 24-cell with congruent copies of a single Hamiltonian cycle, using the algebra of quaternions.

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1. AROUND THE WORLD. In modern terms, a Hamiltonian cycle is a path on a graph, which passes through every vertex exactly once before returning to its starting point. Hamilton's name is attached to the idea because of his "Icosian Game", a puzzle that amounted to finding such cycles among the vertices and edges of a dodecahedron [**2**].

Anyone with a few spare minutes can verify that the graphs of the five Platonic solids all contain Hamiltonian cycles. The octahedron, however, is unique among the five in one interesting respect. The cycle shown in Figure 1 uses exactly half of the edges in the graph, and can be rotated 90 degrees to give a congruent copy of the cycle that covers the other half of the edges. Thus, the edges of the octahedron can be covered by disjoint, geometrically congruent copies of a single Hamiltonian cycle; briefly, the octahedron can be *tiled* by one of its Hamiltonian cycles. None of the other Platonic solids can be tiled in this way. This is simply a matter of divisibility; for such a tiling to exist, the number of edges must be divisible by the number of vertices.

In this note, we show how to construct a tiling Hamiltonian cycle on the edges of the 24-cell, one of the six regular convex polytopes in dimension four [**5**]. Finding such a cycle is within the realm of brute-force computation [**6**], so the interest here is in the construction, which is algebraic and can be verified by hand. We begin with an overview of the 24-cell's structure and the particular features that make our construction possible.

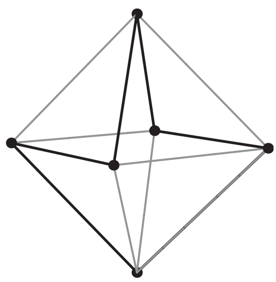

Figure 1. Hamiltonian cycle that tiles the edges of the octahedron

2. STRUCTURE OF THE 24-CELL. Combinatorially, the 24-cell consists of 24 vertices, with 8 edges meeting at each vertex for a total of 96 edges. It can be constructed by gluing the triangular faces of 24 octahedra together in pairs. Concretely, we can situate the 24-cell in $\mathbb{R}^4$ with 16 vertices at $(\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$ and additional vertices at all 8 permutations of $(\pm 1, 0, 0, 0)$. Vertices are connected by an edge if they are unit distance apart. These are the vertices of a 4-dimensional hypercube together with the vertices of its dual "cross polytope", and it is peculiar to 4-dimensional geometry that taking the cube and its dual together in this way results in a *regular* polytope [**4**].

Besides the simplicity of the coordinates, this particular embedding of the 24-cell has algebraic significance. If we identify the vertex at (a, b, c, d) with the quaternion $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, then the vertices form a nonabelian group under quaternion multiplication, which we will call G. This 24-element group is known as the *binary tetrahedral group* [**3**].

Adjacency of vertices can be characterized algebraically. Define

$$g_1 = (1 + \mathbf{i} + \mathbf{j} - \mathbf{k})/2,$$
$$g_2 = (1 + \mathbf{i} - \mathbf{j} + \mathbf{k})/2,$$
$$g_3 = (1 - \mathbf{i} + \mathbf{j} + \mathbf{k})/2,$$

and

$$g_4 = (1 - \mathbf{i} - \mathbf{j} - \mathbf{k})/2.$$

The g_i's and their inverses (their quaternion conjugates) are one unit away from the identity element. It follows that the neighbors of any vertex $v \in G$ are $vg_1, vg_1^{-1}, \ldots, vg_4, vg_4^{-1}$ (since multiplication by a unit quaternion is an isometry), and if $vg_i^{\pm 1} = w$, then we say that there is an edge of type g_i between v and w. From an "edge list" $E = e_1e_2e_3\cdots$ with each $e_i \in \{g_1^{\pm 1}, g_2^{\pm 1}, g_3^{\pm 1}, g_4^{\pm 1}\}$, we can construct an asssociated path $P = v_0v_1v_2\cdots$ by defining $v_0 = 1$ and $v_i = v_{i-1}e_i$ for $i \geq 1$. This is how we will present our tiling Hamiltonian cycle (in fact using only g_1 and g_2).

If we define the *class* of a vertex (a, b, c, d) to be $+1$, -1, or 0, according to the sign of the product $abcd$, then the vertices of class 0 form a quaternion subgroup of order 8 that is normal in G. The $+1$ and -1 classes are the other two cosets of this normal Q_8, and the class function is in fact a homomorphism from G to $\mathbb{Z}/3\mathbb{Z}$. A nice feature of the cycle we will construct is that it visits the three classes in turn, periodically; this will be evident in the construction.

3. CONSTRUCTING THE CYCLE. To begin the construction, let $\theta : \mathbb{R}^4 \to \mathbb{R}^4$ by $\theta(w, x, y, z) = (w, x, -z, y)$. This rotation θ is an order 4 symmetry of the 24-cell, which will play the role of a tiling symmetry analogous to 90-degree rotation about the z-axis in Figure 1. The restriction of θ to G is a group automorphism with $\theta(\mathbf{j}) = \mathbf{k}$ and $\theta(\mathbf{k}) = -\mathbf{j}$.

If $P = v_0v_1v_2\cdots$ is the path associated to an edge list $E = e_1e_2e_3\cdots$, we let θP denote the path $\theta(v_0)\theta(v_1)\theta(v_2)\cdots$. Since θ is a group automorphism, the edge list for θP is $\theta E = \theta(e_1)\theta(e_2)\theta(e_3)\cdots$. We can apply θ again to obtain successive rotations $\theta^2 P$ and $\theta^3 P$.

Let $K = \langle \mathbf{k} \rangle$, a cyclic subgroup of order 4. K has 6 right cosets in G, and, since $K \leq Q_8$, elements in the same coset of K also belong to the same vertex class. Our first result is a calculation, which is easy to check.

Lemma 1. *θ^2 acts on right cosets of K by exchanging the two cosets in the -1 class, and leaving the others fixed.*

For the remaining results, let $E = \overline{e_1e_2e_3e_4e_5e_6}$ denote an edge list with the six e_i's repeating periodically, and let $P = v_0v_1v_2\cdots$ be the associated path. The next two results give simple algebraic criteria for the e_i's, which ensure that P is a Hamiltonian cycle and tiles the 24-cell under θ.

Lemma 2. *If $v_0, v_1, \ldots, v_5$ belong to distinct cosets of K and $e_1e_2\cdots e_6 = \mathbf{k}$, then $v_0, \ldots, v_{23}$ are all distinct, and constitute a Hamiltonian cycle.*

Proof. From $e_1e_2\cdots e_6 = \mathbf{k}$ and the periodic structure, it follows that $v_{r+6} = \mathbf{k}v_r$ for each r. Thus, the six sets $\{v_r, v_{r+6}, v_{r+12}, v_{r+18}\}$, $0 \leq r < 6$, are precisely the six cosets of K in G, and $v_{24} = \mathbf{k}^4 v_0 = v_0$ (so P returns to its starting point once it has visited all the vertices). ■

Lemma 3. *Suppose, in addition to the conditions of Lemma 2, that $e_i \in \{g_1, g_2\}$ for each i, and $e_2 = e_5$. Then P, θP, $\theta^2 P$, and $\theta^3 P$ are pairwise disjoint (have no edges in common).*

Proof. First, note that g_1 and g_2 both have class -1, so the path P cycles periodically through the three classes 0, -1, and 1.

Next, observe that $\theta(g_1) = g_4^{-1}$ and $\theta(g_2) = g_3^{-1}$, so P, which has only g_1 and g_2 edges, cannot have any edges in common with θP or $\theta^3 P$. But $\theta^2(g_1) = g_2$ and $\theta^2(g_2) = g_1$, so it seems feasible that P and $\theta^2 P$ might have some edges in common. Consider any vertex v_r in the path P, and suppose that $\theta^2(v_r) = v_s$. The edge leaving v_r in P is of type e_{r+1} (either g_1 or g_2), and so the edge leaving v_s in $\theta^2 P$ is the other type ($\theta^2(e_{r+1}) \neq e_{r+1}$).

If $r \equiv 0, 2, 3, \text{ or } 5 \pmod 6$, then the class of v_r is either 0 or 1, so $\theta^2(v_r)$ is in the same coset of K as v_r is (by Lemma 1). It follows that $r \equiv s \pmod 6$, and so $e_{r+1} = e_{s+1}$. Thus the edge leaving v_s in P has type e_{r+1} and does not coincide with the edge leaving v_s in $\theta^2 P$.

If $r \equiv 1 \text{ or } 4 \pmod 6$, then v_r is in one of the class -1 cosets and v_s is in the other. In this case, our hypothesis that $e_2 = e_5$ ensures that the edge leaving v_s in P differs from the edge leaving v_s in $\theta^2 P$. ■

We may now present our tiling cycle.

Corollary. *The path associated to $E = \overline{g_1 g_2 g_2 g_1 g_2 g_1}$ is a Hamiltonian cycle which tiles the 24-cell under θ.*

It is only a matter of arithmetic to check that these six terms satisfy the conditions of the two preceding lemmas. Finding this sequence was not difficult, since the conditions of the lemmas are so restrictive that once the first term is chosen, the remaining five follow from it uniquely.

Hamilton is said to have been "inordinately proud" of his Icosian Game [**2**]. This was probably not due to its sheer entertainment value, but rather because the solution to various cycle-finding problems on the dodecahedron could be interpreted in terms of his noncommutative Icosian Calculus [**1**]. He might well have been pleased that his earlier noncommutative discovery, the algebra of quaternions, could be used in much the same way to steer us about such a beguiling object as the 24-cell.

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Department of Mathematics, Washington & Lee University, Lexington, VA 24450
siehlerj@wlu.edu