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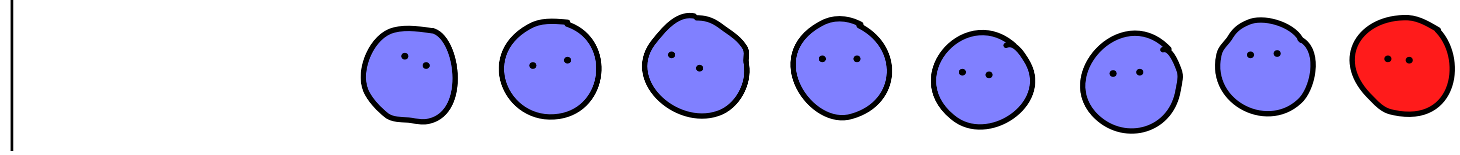

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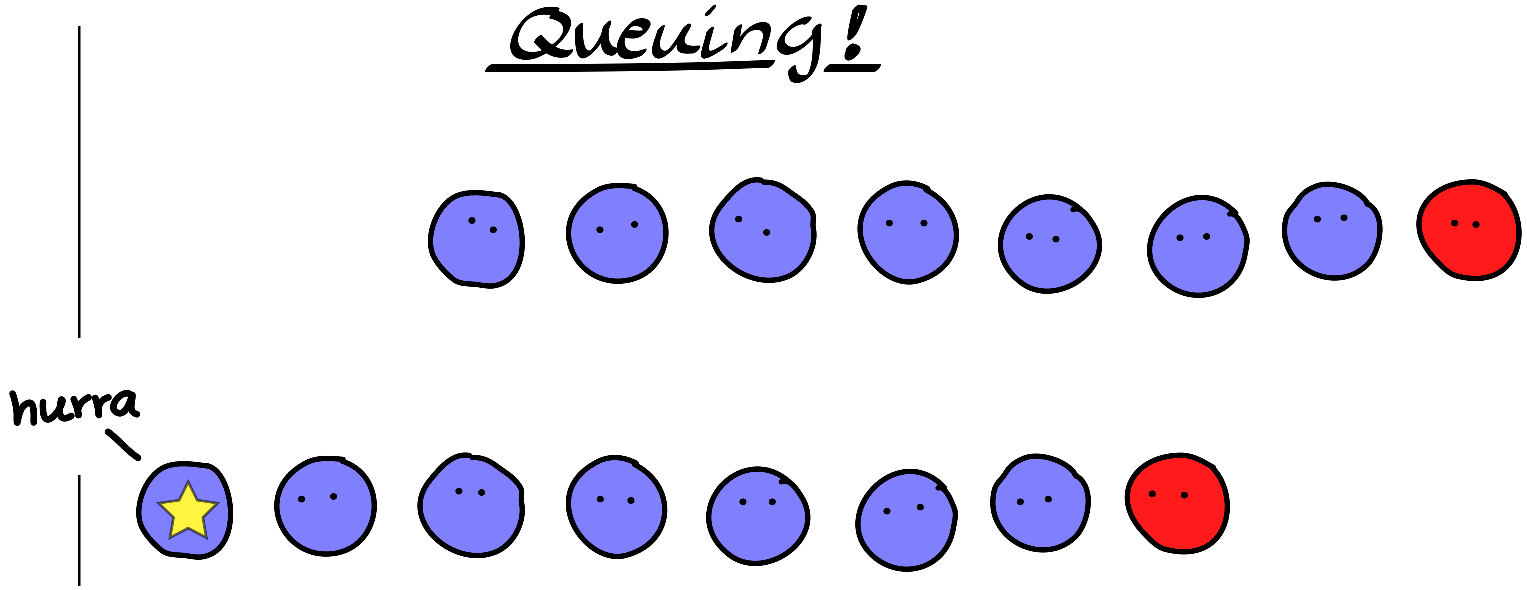

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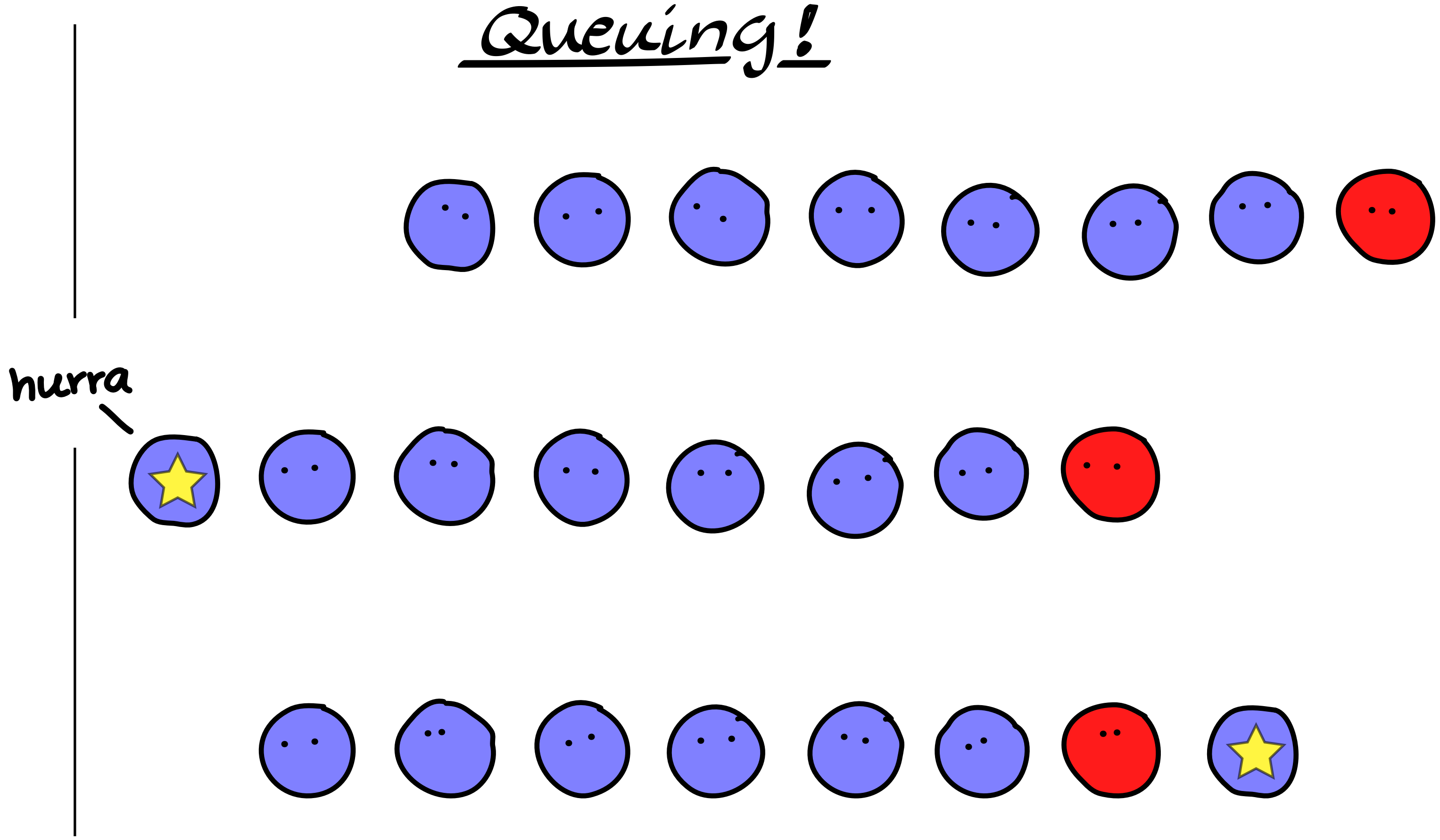

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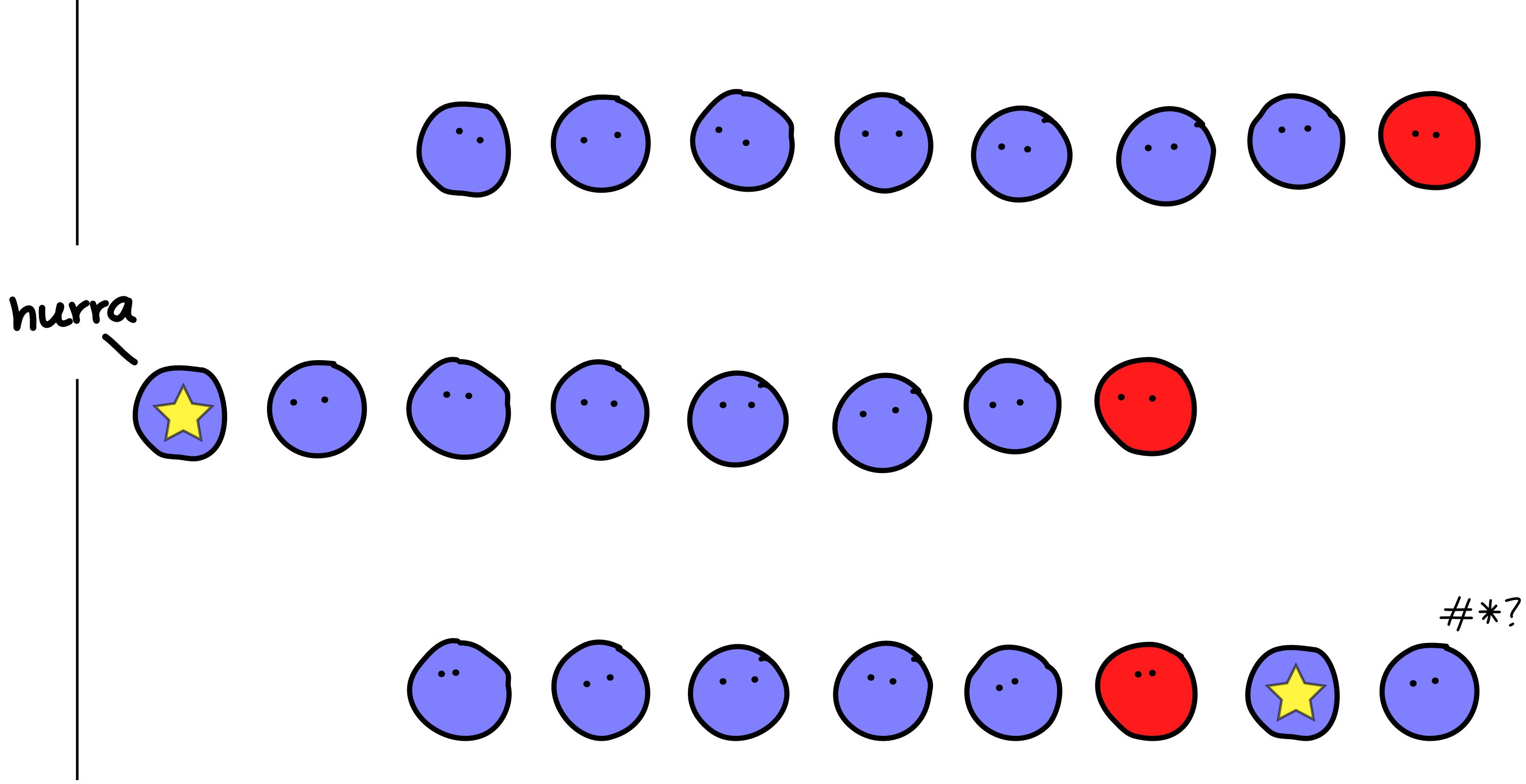

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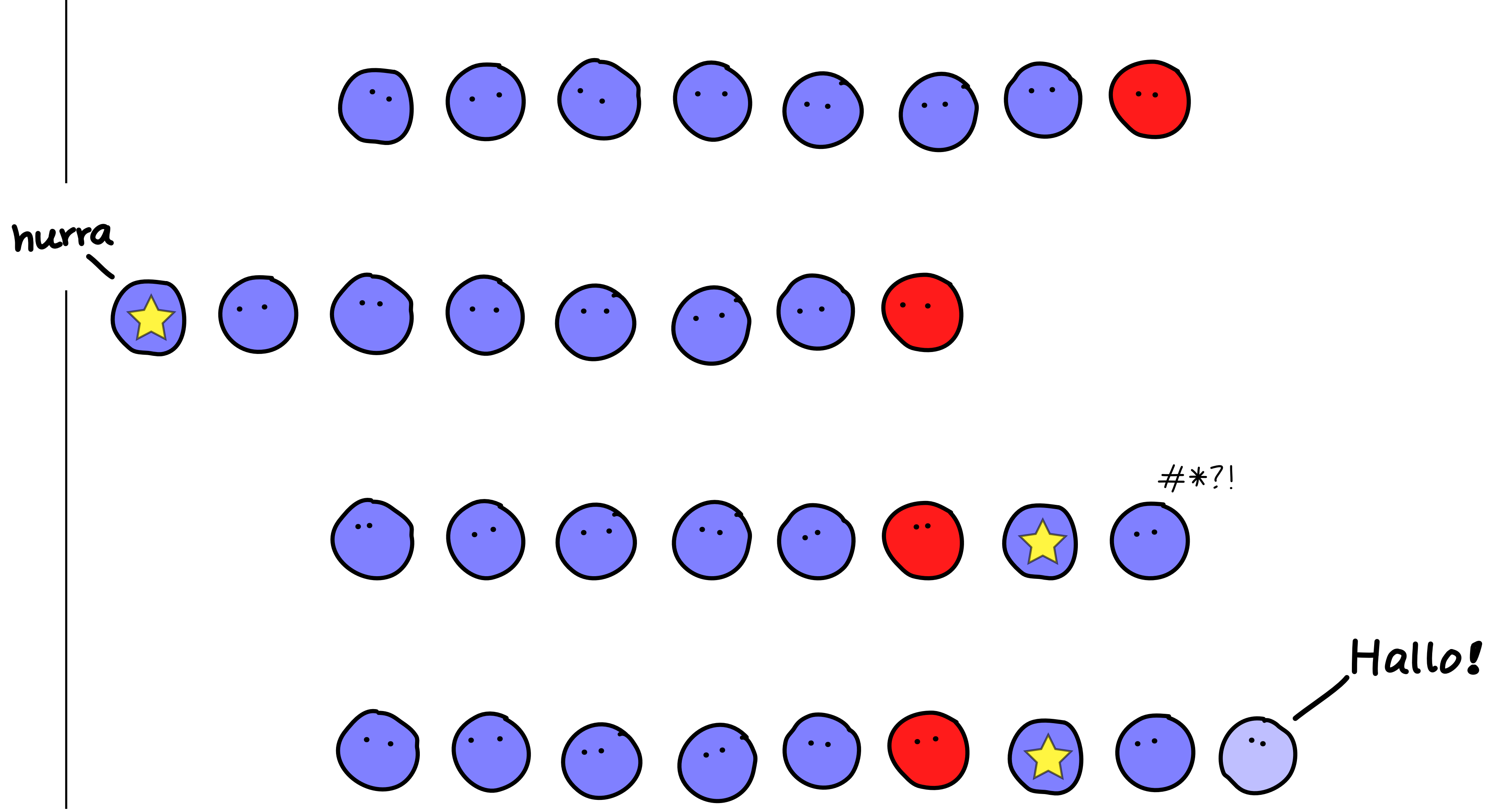

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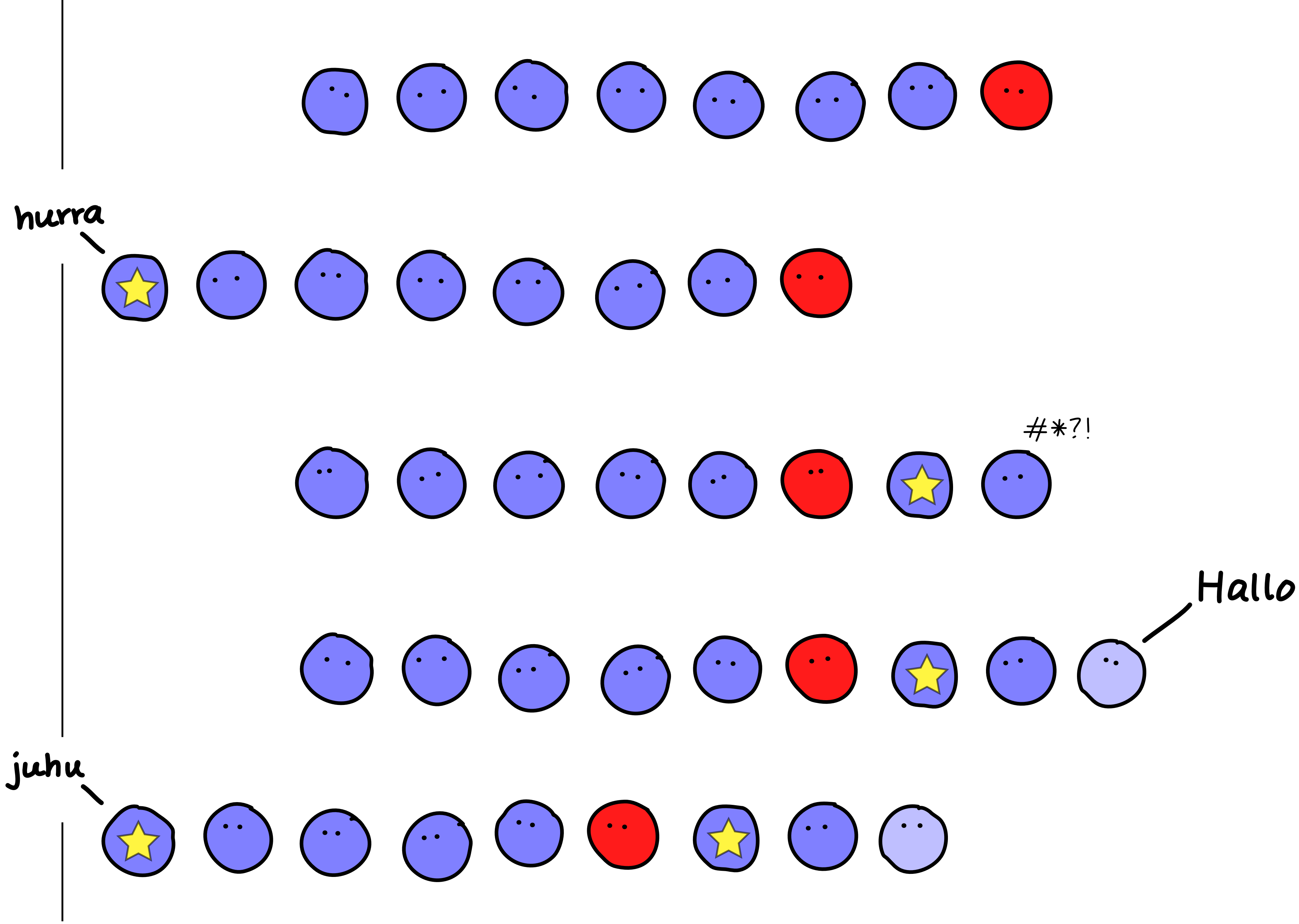
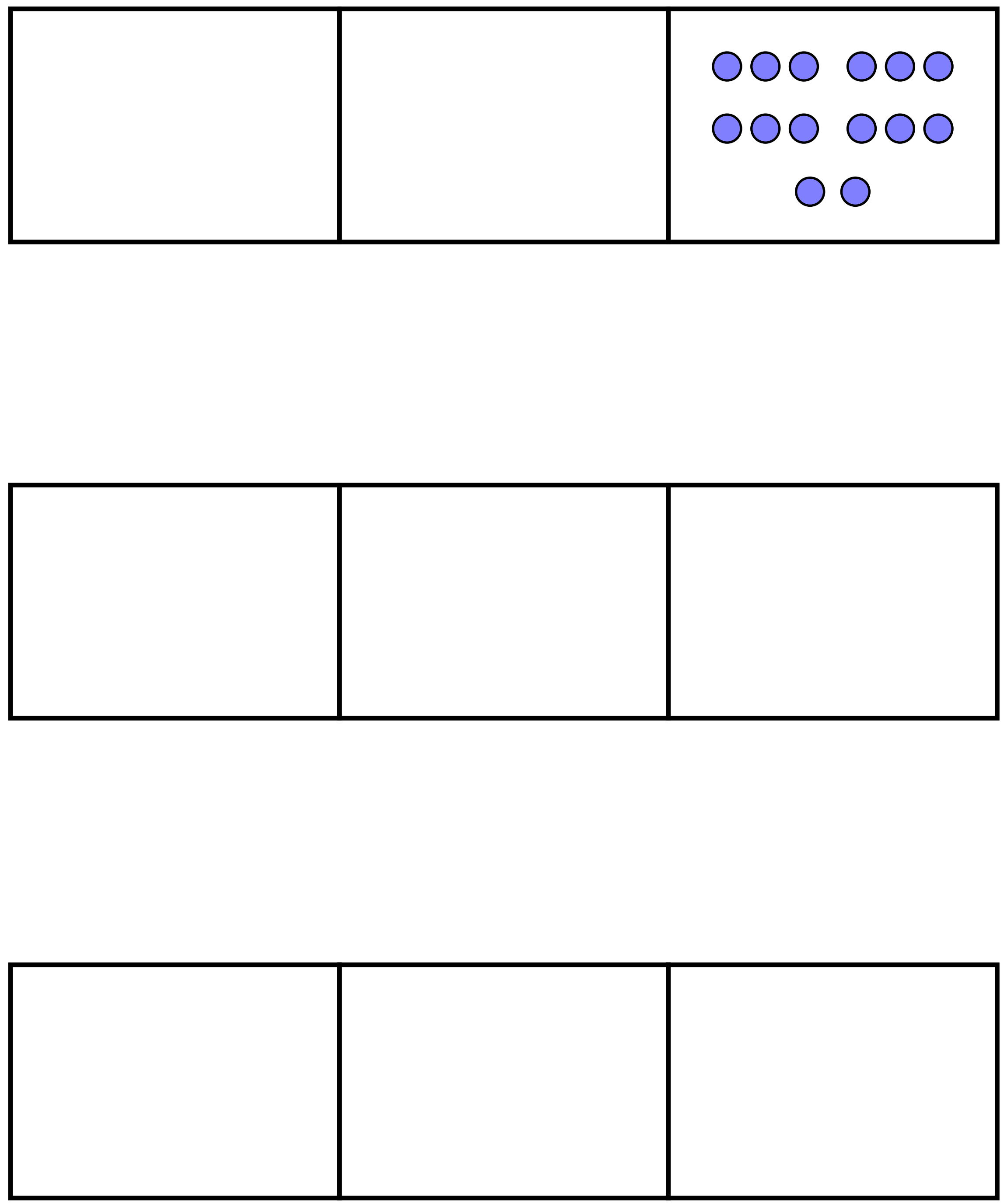


u.s.w.

Ex: Base ●●●

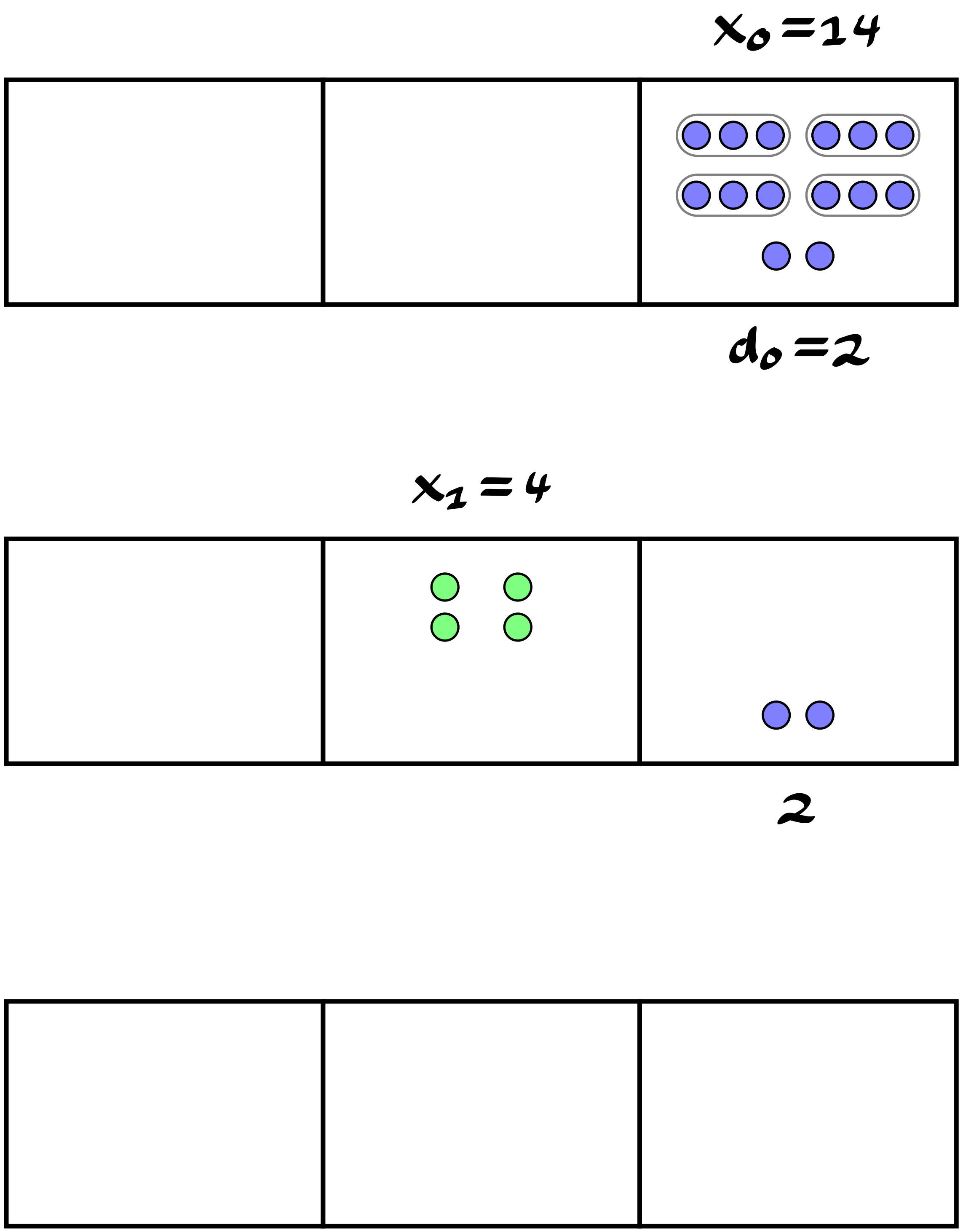
Digit Algorithm

$x_0 = 14$



$$\begin{cases} x_n = \lfloor x_{n-1}/3 \rfloor \\ d_n = x_n \bmod 3 \end{cases}$$

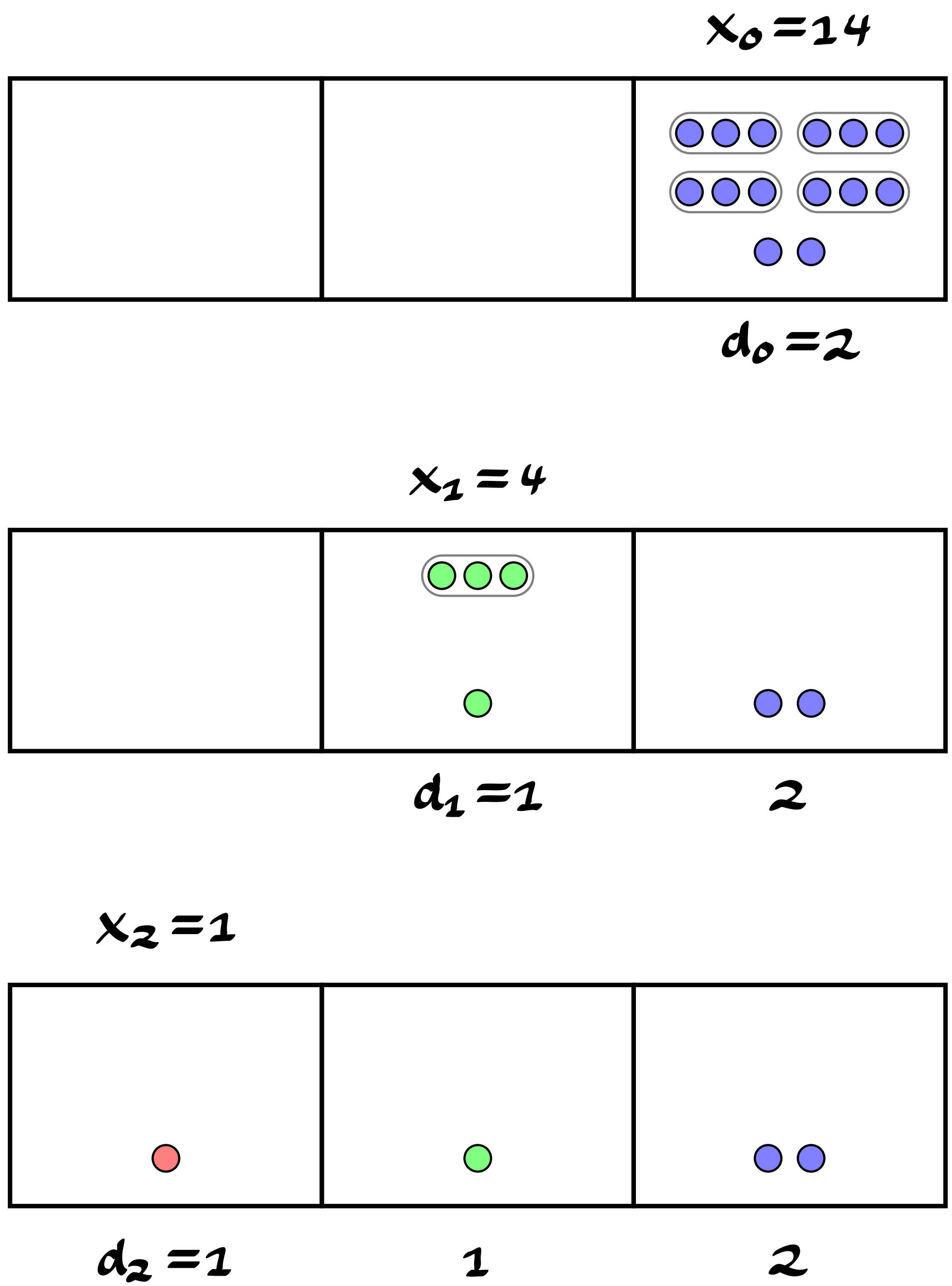
Ex: Base ●●●



Digit Algorithm

$$\begin{cases} x_n = \lfloor x_{n-1}/3 \rfloor \\ d_n = x_n \bmod 3 \end{cases}$$

Ex: Base ●●●



Digit Algorithm

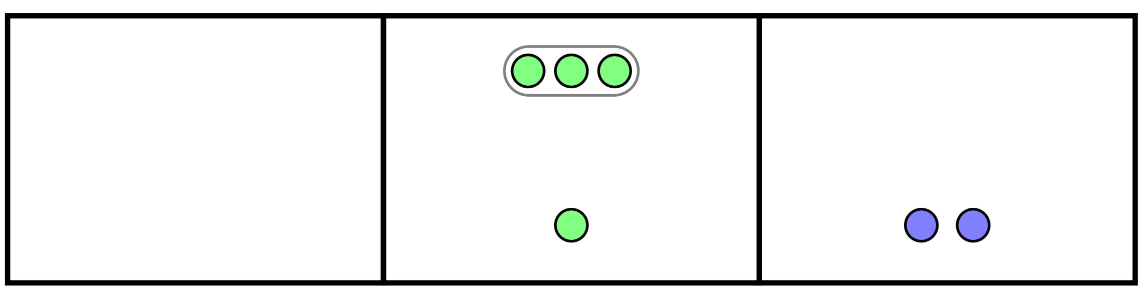
$$\begin{cases} x_n = \lfloor x_{n-1} / 3 \rfloor \\ d_n = x_n \bmod 3 \end{cases}$$

Ex: Base ●●●

$x_0 = 14$

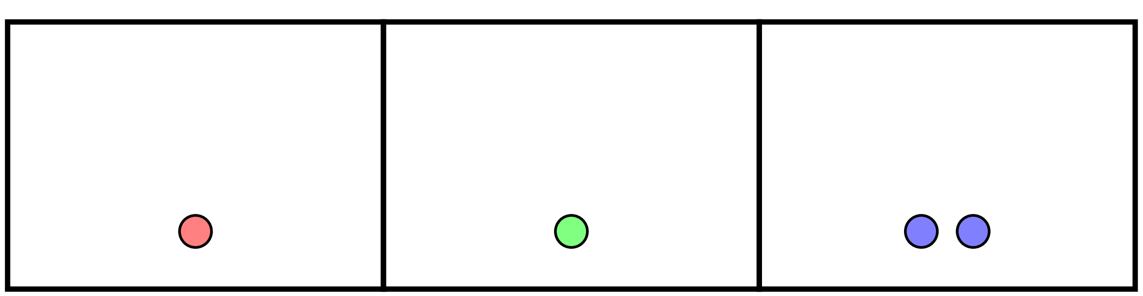
$d_0 = 2$

$x_1 = 4$



$d_1 = 1$ 2

$x_2 = 1$



$d_2 = 1$ 1 2

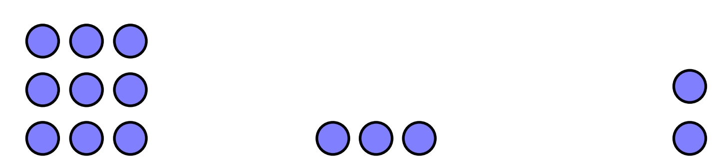
Digit Algorithm

$$\begin{cases} x_n = \lfloor x_{n-1} / 3 \rfloor \\ d_n = x_n \bmod 3 \end{cases}$$

Value formula

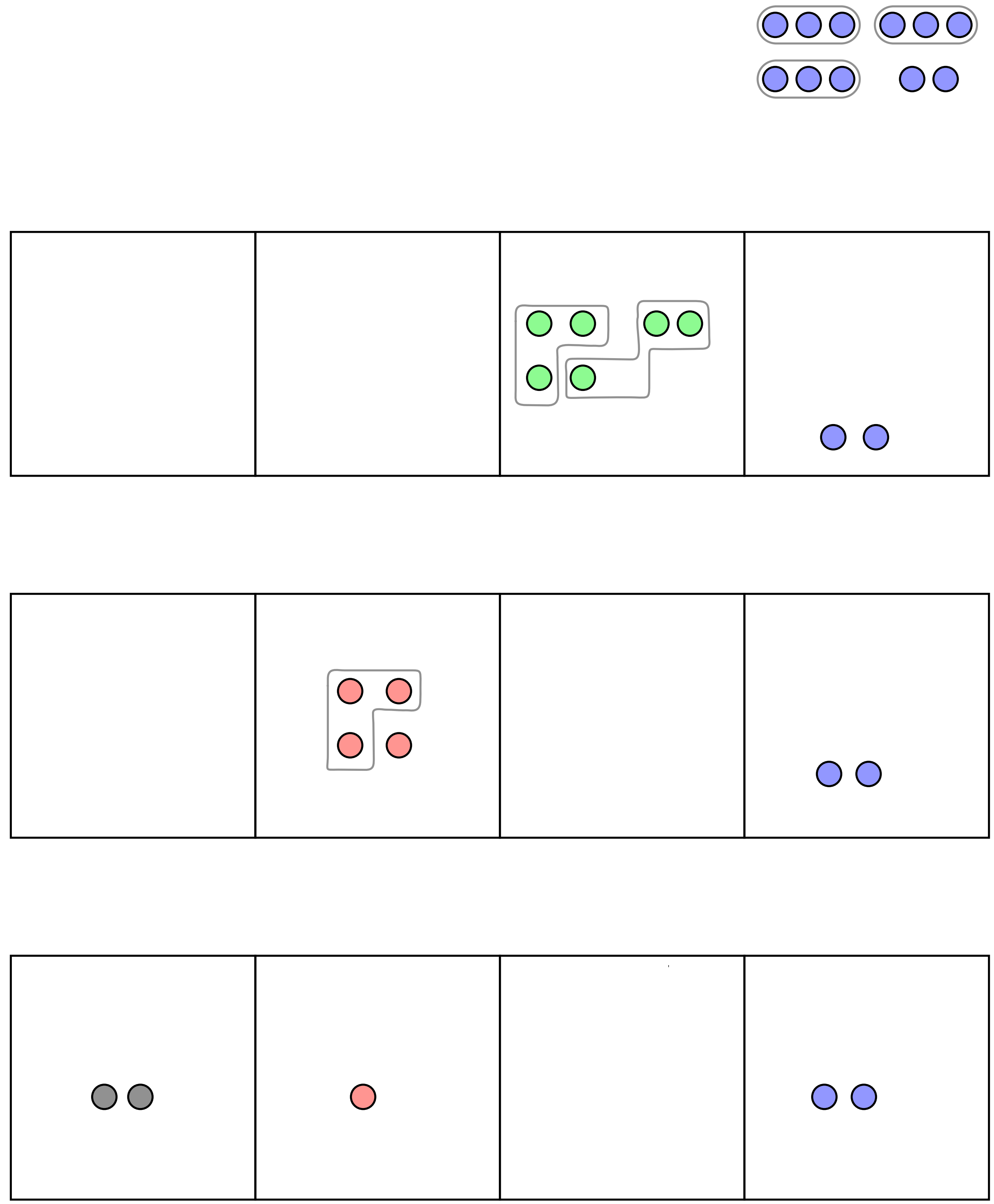
$$\sum d_i 3^i$$

$$= 1 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0$$



Ex Base ●●●:●●

"base 3/2"

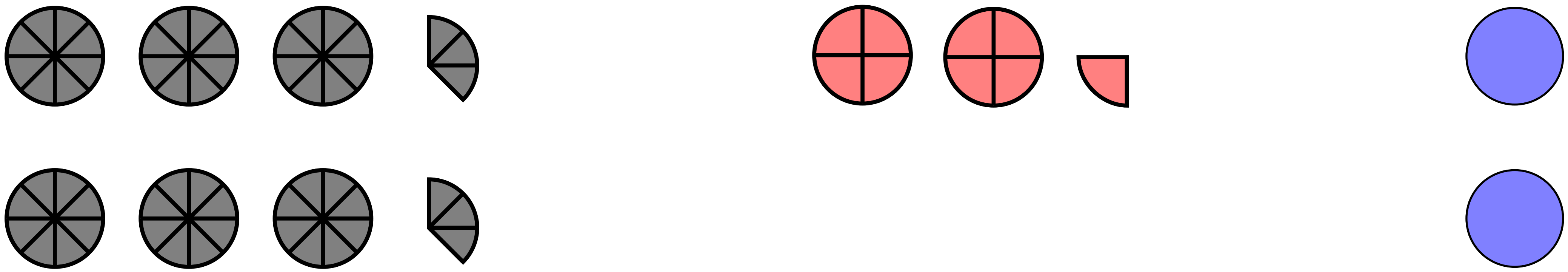


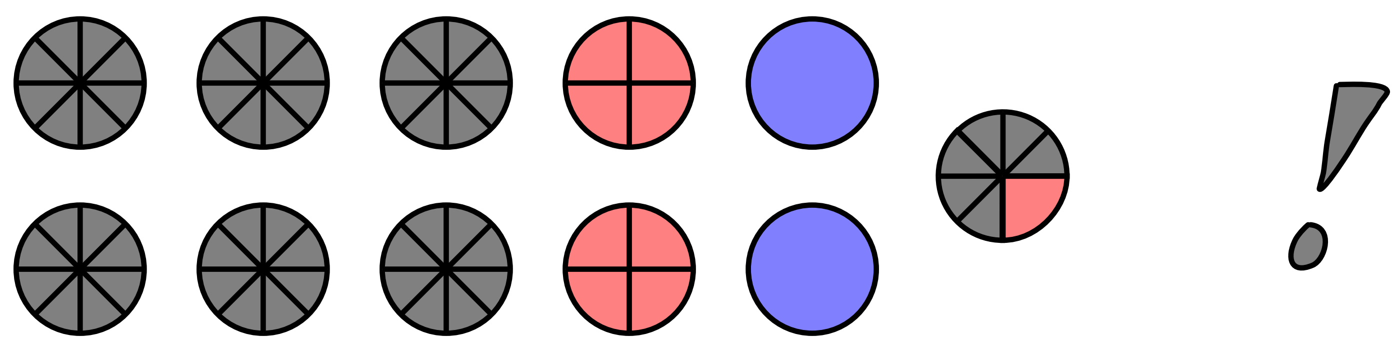
Ex Base

"base 3/2", i.y.l.

2	1	0	2

$2 \cdot 27/8$ $1 \cdot 9/4$ $2 \cdot 1$





Digits & Values
summary

Base p/q Digit Algorithm

$$x_0 = n$$
$$x_j = q\lfloor x_{j-1}/p \rfloor$$
$$d_j = x_j \% p$$

$n \longrightarrow \{d_j\}$

Value Formula

$$n = \sum_j (p/q)^j d_j$$

$\{d_j\} \longrightarrow n$

Ex: Base 10/3

$$x_0 = 49 \qquad d_0 = 49 \% 10 = 9$$

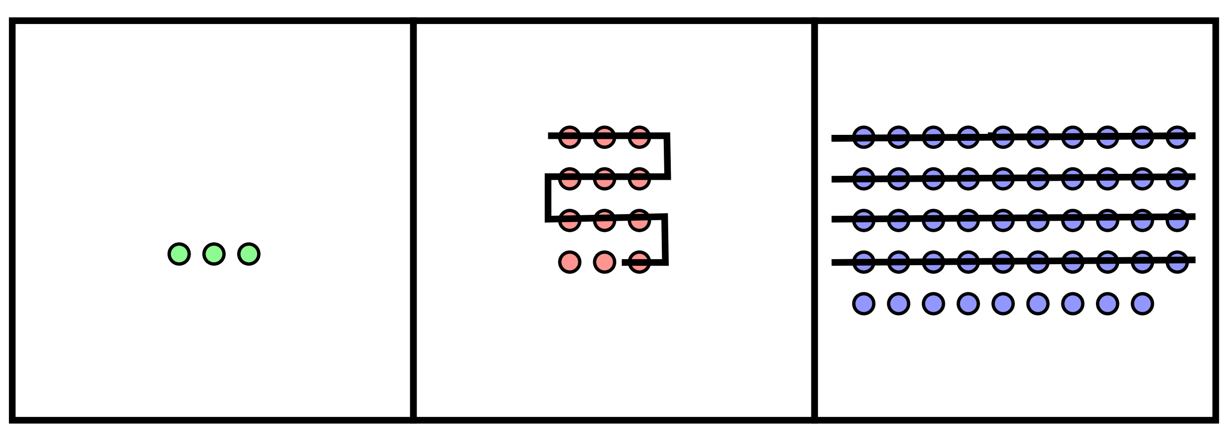
$$x_1 = 3\lfloor 49/10 \rfloor = 12 \qquad d_1 = 12 \% 10 = 2$$

$$x_2 = 3\lfloor 12/10 \rfloor = 3 \qquad d_2 = 3 \% 10 = 3$$

$$x_3 = x_4 = \cdots = 0$$

$$3(10/3)^2 + 2(10/3) + 9 = \frac{100 + 20 + 27}{3}$$

$$= 49 \checkmark$$



Ex: Base 2/3

what could possibly go wrong?

$x_0 = 15$

$d_0 = 15 \% 2 = 1$

Ex: Base 2/3

what could possibly go wrong?

$x_0 = 15$

$x_1 = 3\lfloor 15/2 \rfloor = 21$

$d_0 = 15 \% 2 = 1$

$d_1 = 21 \% 2 = 1$

Ex: Base 2/3

what could possibly go wrong?

$x_0 = 15$

$x_1 = 3\lfloor 15/2 \rfloor = 21$

$x_2 = 3\lfloor 21/2 \rfloor = 30$

$d_0 = 15 \% 2 = 1$

$d_1 = 21 \% 2 = 1$

$d_2 = 30 \% 2 = 0$

Ex: Base 2/3

what could possibly go wrong?

$x_0 = 15$ $\qquad d_0 = 15 \% 2 = 1$

$x_1 = 3\lfloor 15/2 \rfloor = 21$ $\qquad d_1 = 21 \% 2 = 1$

$x_2 = 3\lfloor 21/2 \rfloor = 30$ $\qquad d_2 = 30 \% 2 = 0$

$x_3 = 3\lfloor 30/2 \rfloor = 45$ $\qquad d_3 = 45 \% 2 = 1$

$x_4 = 3\lfloor 45/2 \rfloor = 66$ $\qquad d_4 = 66 \% 2 = 0$

$x_5 = 3\lfloor 66/2 \rfloor = 99$ $\qquad d_5 = 99 \% 2 = 1$

$x_6 = 3\lfloor 99/2 \rfloor = 147$ $\qquad d_6 = 147 \% 2 = 1$

So...

$$15 = 1101011\ldots ?$$

finite

Infinite digit sequences?

$$n=15$$

base 2/3 digit algorithm:

$$\{d_j\} = 1, 1, 0, 1, 0, 1, 1, \ldots$$ seems strange

Perhaps this is fine:

Infinite digit sequences?

$$n = 15$$

base 2/3 digit algorithm:

$$\{d_j\} = 1, 1, 0, 1, 0, 1, 1, \ldots$$

seems strange

Perhaps this is fine:

$$\sum (2/3)^j d_j = 1 + 1\cdot(2/3)^1 + 0\cdot(2/3)^2 + 1\cdot(2/3)^3 + \ldots$$

$$< 1 + 1\cdot(2/3)^1 + 1\cdot(2/3)^2 + 1\cdot(2/3)^3 + \ldots$$

$$= \frac{1}{1 - 2/3} = 3$$

Infinite digit sequences?

$$n = 15$$

base 2/3 digit algorithm:

$$\{d_j\} = 1, 1, 0, 1, 0, 1, 1, \ldots$$ seems strange

Perhaps this is fine:

$$\sum (2/3)^j d_j = 1 + 1\cdot(2/3)^1 + 0\cdot(2/3)^2 + 1\cdot(2/3)^3 + \ldots$$

$$< 1 + 1\cdot(2/3)^1 + 1\cdot(2/3)^2 + 1\cdot(2/3)^3 + \ldots$$

$$= \frac{1}{1 - 2/3} = 3$$

The value formula $n = \sum_j (p/q)^j d_j$ cannot be true (when $p/q < 1$)

The value formula $n = \sum_j (p/q)^j d_j$ is perfectly fine even when $p/q < 1$

p-Norm and Distance

prime factorization in $\mathbb{Q}$:

$$\frac{24}{35} = 2^3 \times 3 \times 5^{-1} \times 7^{-1} \qquad \frac{121}{5000} = 2^{-4} \times 5^{-3} \times 11^2$$

2-norms:

$$\left|\frac{24}{35}\right|_2 = \frac{1}{2^3} \qquad \left|\frac{121}{5000}\right|_2 = \frac{1}{2^{-4}} = 16$$

11-norms:

$$\left|\frac{24}{35}\right|_{11} = \frac{1}{11^0} = 1 \qquad \left|\frac{121}{5000}\right|_{11} = \frac{1}{11^2}$$

etc.

special definition for all p $\quad |0|_p = 0$

p-Distance and Convergence

p-distance: $d_p(q, r) = |q-r|_p \longrightarrow$ metric space $\mathbb{Q}_p$

Ex. $d_2(30, 6) = |24|_2 = \frac{1}{2^3} = \frac{1}{8}$

00110

11110

(agree to three places)

Ex. $d_p\left(\left(\frac{p}{p+1}\right)^{30}, 0\right) = \frac{1}{p^{30}}$

thus

Ex. $1, 3, 7, 15, 31, \ldots \longrightarrow -1$ in $\mathbb{Q}_2$*

Ex. $\left\{\left(\frac{p}{q}\right)^k\right\} \longrightarrow 0$ in $\mathbb{Q}_p$

least terms

* so $\sum_{k=0}^{\infty} 2^k = \frac{1}{1-2} = -1$ in $\mathbb{Q}_2$

Partial sums

setup

Base p/q digit algorithm: $n \to \{x_j\}$ quotients, $\{d_j\}$ remainders

Convergence lemma

Let $y_k = \sum_{j=0}^{k-1} (p/q)^j d_j$.

proof – routine induction on k

Then $(p/q)^k \cdot x_k = n - y_k \quad \forall k > 0$.

integer

Consequently,

- $p^k \mid n - y_k$ and so $|n-y_k|_p < 1/p^k$
- $y_k \to n$ in $\mathbb{Q}_p$

The value formula

setup

Base p/q digit algorithm: $n \to \{x_j\}$ quotients, $n \to \{d_j\}$ remainders

Theorem

$$n = \sum_j (p/q)^j d_j ,$$

with the series converging in $\mathbb{Q}_p$ when $p/q < 1$

Example

$$2018 \xrightarrow{\text{base } 3/7} \{d_j\} = 2, 0, 2, 1, 0, 1, \ldots$$

so $2 + 0(3/7) + 2(3/7)^2 + 1(3/7)^3 + 0(3/7)^4 + 1(3/7)^5 + \ldots$

$= 2018$ in $\mathbb{Q}_3$

Periodicity

Ex: base 3/4

$$n = 7 \to \{x_j\} = 7 \quad 8 \quad 8 \ \ldots$$
$$n = 7 \to \{d_j\} = 1 \quad 2 \quad 2 \ \ldots$$

$$n = 8 \to \{x_j\} = 8 \quad 8 \quad 8 \ \ldots$$
$$n = 8 \to \{d_j\} = 2 \quad 2 \quad 2 \ \ldots$$

$$n = 9 \to \{x_j\} = 9 \quad 12 \quad 16 \quad 20 \quad 24 \quad 32 \quad 40 \ \ldots$$
$$n = 9 \to \{d_j\} = 0 \quad 0 \quad 1 \quad 2 \quad 0 \quad 2 \quad 1 \ \ldots$$

how long is the period?

Periodicity

Theorem

Let $n \to \{d_j\}$ in base $p/q < 1$.

If $\{d_j\}$ is periodic, then $n \leq \dfrac{q(p-1)}{q-p}$.

periodicity threshold

Ex. $p/q = 3/4$

$\{d_j\}$ periodic $\Rightarrow n \leq \dfrac{4 \cdot 2}{1} = 8$,

and $n > 8 \Rightarrow \{d_j\}$ not periodic.

Theorem

Let $n \to \{d_j\}$ in base $p/q < 1$.
If $\{d_j\}$ is periodic, then $n \leq \dfrac{q(p-1)}{q-p}$.

Simplified proof (a single repeating digit)

Suppose $n \longrightarrow \{d_j\} = d,\ d,\ d,\ \ldots$

$$\text{Then } n = \sum_{j=0}^{\infty} d(p/q)^j \text{ in } \mathbb{Q}_p$$

$$= \lim_{k\to\infty} d + d(p/q) + \cdots + d(p/q)^{k-1} \text{ in } \mathbb{Q}_p$$

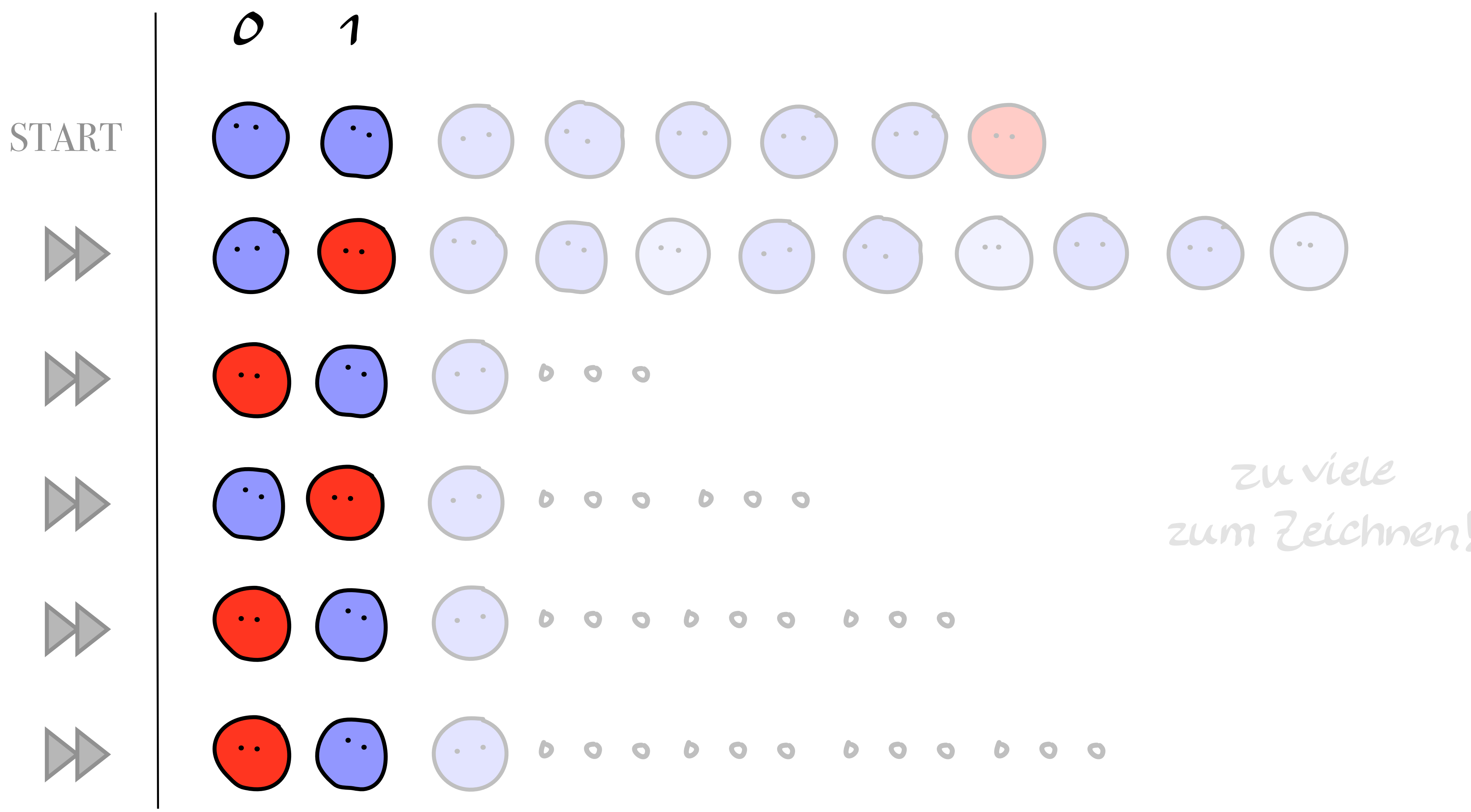
$$= \lim_{k\to\infty} \frac{d\left(1-(p/q)^k\right)}{1-p/q}$$

0 in $\mathbb{Q}_p$

max digit

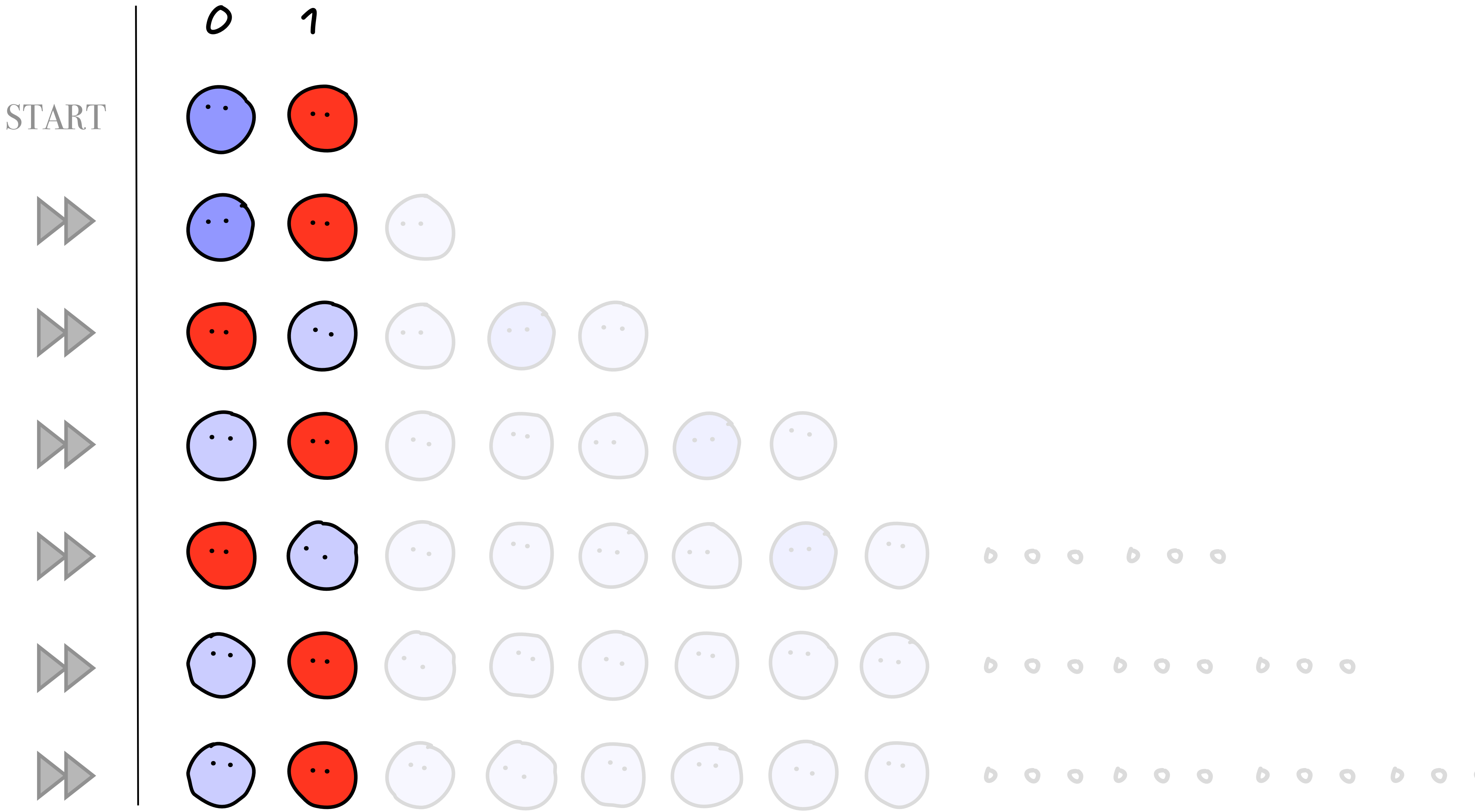
$$= \frac{d}{1-p/q} \leq \frac{p-1}{1-p/q} = \frac{q(p-1)}{q-p}. \quad \blacksquare$$

Q_sequences



$$1, 0, 1, 0, 0, 1, \ldots \quad = \quad Q(8)$$

Q_sequences



$$1, 1, 0, 1, 0, 1, 1, \ldots = Q(2)$$

Theorem Q

$Q(n)$ is the base $2/3$ expansion of $3(3n-1)$.

Ex. $n=2$

$$Q(2) = 1,1,0,1,0,1,1,\ \dots$$

while

$$3(3\cdot 2-1) = 15 \longrightarrow 1,1,0,1,0,1,1,\ \dots \text{ in base } 2/3$$

Corollary:

$Q(n)$ is aperiodic for every n.

Proof

$$n \geq 1 \Rightarrow 3(3n-1) \geq 6$$

while $$\frac{3(2-1)}{3-2} = 3.$$

periodicity threshold for base 2/3

Corollary

$Q(n)$ contains infinitely many 0's, for every n.